Volatility Risks and Growth Options

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November 7, 2013

Abstract

We propose to measure growth opportunities by firms’ exposure to idiosyncratic volatility news. Theoretically, we show that the value of a growth option increases in idiosyncratic volatility but its response to volatility of aggregate shocks can be either positive or negative depending on option moneyness. Empirically, we show that price sensitivity to variation in idiosyncratic volatility carries significant information about firms’ future investment and growth even after controlling for conventional proxies of growth options such as book-to-market and other relevant firm characteristics. Consistent with our theoretical arguments, we also find that firms’ exposure to aggregate volatility, while priced, does not help predict their future growth. Option-intensive firms identified using our idiosyncratic volatility-based measure earn a lower premium than do firms that rely more heavily on assets in place.

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Introduction

We propose a volatility-based measure of growth options owned by firms. Our idea originates from the conventional wisdom that option payoffs increase in volatility of the underlying cash flows. According to the standard real option models that typically make no distinction between aggregate and idiosyncratic risks, price exposure to volatility news should contain information about growth opportunities available to firms. We argue that the source of variation does matter for this result to hold. We show that the value of a growth option increases with idiosyncratic volatility but its response to volatility of aggregate shocks is ambiguous. Guided by our theoretical analysis, we propose to measure options by exposure to idiosyncratic volatility news. We show that, empirically, it carries significant information about cross-sectional differences in firms’ future investment and growth even after controlling for conventional proxies of growth options. We also find that option-intensive firms, identified by high sensitivity to idiosyncratic volatility, earn a lower premium compared with low-exposure asset-in-place intensive firms.

Unobservable growth opportunities are typically proxied by various valuation ratios such as market-to-book, price-to-earnings etc. Valuation-based proxies, however, have significant limitations as they may vary across firms for many different reasons. For example, any heterogeneity in firms’ current and/or future productivity in the presence of adjustment costs or differences in riskiness of assets in place will generally lead to cross-sectional differences in market-to-book ratios even in the absence of any growth options. Our volatility-based measure is not subject to these types of biases and, as we show, is supplementary to the traditional proxies.

In the empirical literature and among practitioners, high valuation ratios are commonly associated with high growth-option intensity. Theoretical asset pricing models, however, have different implications for the sign of this relationship. For example, in the models of Berk, Green, and Naik (1999), Gomes, Kogan, and Zhang (2003), and Carlson, Fisher, and Giammarino (2004), market-to-book ratios and growth-option intensity are negatively correlated. Growth options in these models are riskier than assets in places, and to account for the value premium, low market-to-book firms (i.e., value firms) are required to be option
intensive. In Kogan and Papanikolaou (2012), Ai and Kiku (2013), and Ai, Croce, and Li (2013), growth options are less risky than assets in place, and market-to-book ratios and growth options are positively related. Given the very different predictions as to what market-to-book proxies for, a non-valuation based measure of growth options could help us better understand the economic mechanism of the value premium.

Our measure of growth options is motivated by the insight that option payoffs respond positively to volatility news. We argue, however, that the standard intuition applies only to idiosyncratic volatility and cannot be generalized to volatility of aggregate risks. To gain intuition, consider a real option with a fixed strike price. An increase in volatility raises the option value – this is the traditional (partial equilibrium) effect that pertains to all types of volatility. Aggregate uncertainty, however, has an additional discount rate effect. High aggregate volatility lowers the risk-free rate and, hence, the discount rate applied to the strike asset. In a broad class of economic models, high aggregate uncertainty also raises risk premia and, therefore, the discount rate applied to the underlying cash flow. These general equilibrium implications lower the option payoff, working in the opposite direction to the traditional partial equilibrium effect. The overall response of option payoffs to aggregate volatility risks is, therefore, ambiguous and, as we show, depends on moneyness. For options close to the exercise threshold, the general equilibrium effect dominates and option payoffs respond negatively to aggregate volatility shocks. For deep out-of-the-money options, the partial equilibrium effect prevails and option payoffs increase with aggregate volatility.

Our theoretical analysis suggests that the amount of growth options owned by firms is best measured by price sensitivity to idiosyncratic volatility news. Other volatility-based measures (based on aggregate and, hence, total volatility) may be contaminated by the discount rate effect. We, therefore, distinguish between two types of volatility in our empirical work. We measure time-variation in idiosyncratic volatility by variation in firm-level volatility that is orthogonal to fluctuations in aggregate uncertainty. Firm-level volatility and aggregate uncertainty are measured by realized variances of equity returns and returns of the aggregate market portfolio, respectively.¹

We first show that in the data, firm exposure to idiosyncratic volatility shocks (denoted by $\beta^{ID}$) is largely positive, while exposure to aggregate volatility risks ($\beta^A$) is mostly negative. That is, equity prices tend to increase on positive news about idiosyncratic volatility and tend to fall when aggregate uncertainty in the economy is high. The latter is due to the discount rate effect and is consistent with empirical evidence in Bansal, Khatchatrian, and Yaron (2005), and Bansal, Kiku, Shaliastovich, and Yaron (2013). We then show that, controlling for book-to-market and other firm characteristics, firms that are highly sensitive to variation in idiosyncratic volatility are expected to grow and invest at a high rate as they exercise their growth options. In contrast, equity response to aggregate volatility does not appear to be informative about cross-sectional differences in growth opportunities.

Specifically, we document a steep monotonically increasing pattern in sales and investment growth rates across portfolios sorted on exposure to idiosyncratic volatility. The average annual growth in sales almost doubles and average investment growth changes from -0.7% to 8.1% from the bottom to the top quintile portfolios. In addition, firms with high $\beta^{ID}$ are characterized by high R&D spending and high Tobin’s Q, low leverage and low dividend yields, all of which are characteristic of growing firms. For example, the average dividend yield of firms in the top quintile is only 1.5%, whereas it is about 3% for firms in the bottom quintile. Similarly to the value premium, we find that firms with high exposure to idiosyncratic volatility (i.e., option-intensive firms) carry lower premia relative to low-$\beta^{ID}$ firms.

In a regression setting, we show that exposure to idiosyncratic volatility, by itself, is a significant predictor of future investment. Importantly, it provides additional information about firms’ future investment decisions (hence, available growth options) over and beyond the conventional predictors. The effect of idiosyncratic-volatility exposure is both statistically and economically significant after we control for book-to-market, Tobin’s Q, past investment, size and other relevant characteristics. Quantitatively, a one standard deviation increase in $\beta^{ID}$ results in a significant 4% increase in one-year ahead investment growth. Put differently, firms in the top-$\beta^{ID}$ portfolio invest by 10% more than otherwise identical firms in the bottom portfolio do. We also show that, controlling for book-to-market and size, idiosyncratic volatility exposure has a negative effect on expected returns. A one standard
deviation increase in ID-volatility beta lowers the next year return by about 2.2%, on average.

Exposure to aggregate volatility news is also negatively related to expected returns. Firms with large negative $\beta^A$ carry a substantially higher premium compared with zero-exposure firms. Thus, consistent with the long-run risk literature, aggregate volatility risks carry a negative price (Bansal and Yaron (2004)). The difference in average returns between the bottom and top quintile portfolios ranked by $\beta^A$ is about 4% per annum. When in comes to growth options, we find no significant evidence that exposure to aggregate volatility helps predict firms’ future investment. These findings confirm our theoretical argument. The partial-equilibrium effect (that pushes option prices up) and the discount rate effect (that pushes prices down) work against each other, which makes it hard to learn about available growth options by looking at equity exposure to aggregate volatility shocks.

Our theoretical analysis is related to a large literature on (real) option pricing. While a positive effect of idiosyncratic volatility on option payoffs is well established (see Black and Scholes (1973), Merton (1973), McDonald and Siegel (1986), and Dixit and Pindyck (1994)), our paper formalizes this result in a stochastic-volatility model setting and emphasizes the difference between idiosyncratic and aggregate volatility. Several recent studies explore implications of aggregate volatility risks for exchange-traded index options, for example, Bollerslev, Tauchen, and Zhou (2009), and Drechsler and Yaron (2011). Bloom (2009), and Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2011) consider equilibrium implications of volatility shocks in frameworks with irreversible investment.

Our paper is also related to the literature that explores the implications of option exercise and investment decisions for the cross section of asset returns, for example, Berk, Green, and Naik (1999), Gomes, Kogan, and Zhang (2003), Carlson, Fisher, and Giammarino (2004), Zhang (2005), Cooper (2006), Novy-Marx (2007), Garlappi and Yan (2011), Papanikolaou (2011), and Bhamra and Shim (2011) among others. Different from these papers, our focus is on providing a measure of growth options that can be used to test theory. Kogan and Papanikolaou (2012) also propose a theoretically motivated measure of growth opportunities derived in a model that features investment-specific shocks. Our papers complement each other and present corroborative evidence that option-intensive firms carry lower premia than do firms with abundant assets in place.
The rest of the paper is organized as follows. Section 1 provides a theoretical analysis of growth-option exposure to volatility news. We consider two economies that feature variation in either idiosyncratic or aggregate volatility and characterize the dynamics of growth options in each of them. In Section 2 we present evidence that in the data, firms’ exposure to variation in idiosyncratic volatility is informative about firms’ future investment decisions and growth. We also show that, empirically, firms’ exposure to aggregate volatility news does not help identify option-intensive firms. Section 3 provides concluding remarks.

1. Volatility Shocks and Option Returns

In this section, we provide a theoretical analysis of the relationship between option payoffs and volatility shocks. We distinguish between two types of volatility – volatility of aggregate shocks and volatility of idiosyncratic shocks, and characterize the response of growth options to each type of volatility news. We also highlight differences in volatility exposure between growth options and assets in place.

1.1 Setup of the Model

Consider an economy where a representative agent has intertemporal preferences described by the Kreps and Porteus (1978) utility with a constant relative risk aversion parameter, $\gamma$, and a constant intertemporal elasticity of substitution (IES), $\psi$. Time is continuous and infinite. We follow Duffie and Epstein (1992a) and represent preferences as a stochastic differential utility.

We assume that the dynamics of aggregate consumption are described by the following stochastic process:

$$dC_t = C_t [\mu_C dt + \sigma_C (\theta) dB_t],$$

where $\{B_t\}_{t\geq 0}$ is a one-dimensional standard Brownian motion, and $\{\theta_t\}_{t\geq 0}$ is a two-state Markov process with state space $\Theta = \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. The transition probability
of $\theta_t$ over an infinitesimal time interval $\Delta$ is given by

$$\begin{pmatrix} 1 - \lambda_H \Delta & \lambda_H \Delta \\ \lambda_L \Delta & 1 - \lambda_L \Delta \end{pmatrix}. \quad \text{(2)}$$

An asset in place is a project that generates cash flows, $D_t$, that follow:

$$dD_t = D_t \left[ \mu_D dt + \rho \left\{ \sigma_C (\theta) dB_t + \sigma_D (\theta_t) dB^i_t \right\} \right], \quad \text{(3)}$$

where Brownian motion $B_t$ is the aggregate shock that affects consumption and dividends growth simultaneously, and $B^i_t$ is an idiosyncratic shock. The parameter $\rho$ allows the model to account for differences in leverage and volatility of dividend and consumption growth rates. We assume that the project is subject to random termination that arrives at a Poisson rate $\kappa$ per unit of time.

A growth option is the right to obtain the above project by making an irreversible investment of one unit of consumption goods. That is, the owner of the option has the right to adopt the project at any time at a fixed cost before it is terminated. The growth option expires once the project is liquidated.

We consider two special cases of the above general setup:

Case 1. Economy with Idiosyncratic Volatility Shocks:

$$\sigma_C (\theta) = \sigma,$$

$$\sigma_D (\theta) = \theta, \quad \text{for} \quad \theta = \theta_H, \theta_L \quad \text{(4)}$$

Case 2. Economy with Aggregate Volatility Shocks:

$$\sigma_C (\theta) = \theta, \quad \text{for} \quad \theta = \theta_H, \theta_L \quad \text{(5)}$$

$$\sigma_D (\theta) = \sigma.$$

In the economy with idiosyncratic volatility, aggregate volatility is constant and changes
in volatility of dividend growth are purely idiosyncratic. In the economy with aggregate volatility shocks, fluctuations in cash-flow volatility are perfectly correlated with changes in volatility of aggregate consumption. The latter setup allows us to capture the general equilibrium effect of variation in volatility of aggregate shocks. We use the two economies to highlight the equilibrium effect of volatility news on option values.

We make the following assumptions on the parameters of the model.

**Assumptions:** The parameters of the model satisfy:

\[ \beta + \frac{1}{2} \gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta) - \left( 1 - \frac{1}{\psi} \right) \mu_C > 0 \quad \text{for } \theta = \theta_H, \theta_L. \]  

(6)

and

\[ \kappa + \beta + \frac{1}{\psi} \mu_C - \mu_D + \frac{1}{2} \gamma \left( 2 \rho - 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta) > 0 \quad \text{for } \theta = \theta_H, \theta_L. \]  

(7)

As we show in the Appendix, condition (6) ensures that the life-time utility of the agent is finite, and assumption (7) guarantees that the present value of cash flows is finite.

Under the above assumptions, in both economies, the value of assets in place, denoted by \( V_A(\theta, D) \), is a linear function of dividends. That is,

\[ V_A(\theta, D) = a(\theta) D, \]  

(8)

where \( a(\theta) \) is the price-dividend ratio. As we show in the Appendix, in the economy with idiosyncratic volatility shocks, the price-dividend ratio is constant: \( a(\theta_H) = a(\theta_L) \). In the economy with time-varying aggregate volatility, the price-dividend ratio depends on the current state, \( \theta \), and is provided in the Appendix.

We use \( V_O(\theta, D) \) to denote the value of a growth option in state \( \theta \) with current level of dividend \( D \). The value function \( V_O(\theta, D) \) in both economies permits closed form solutions and is detailed in Proposition 1.
Proposition 1 The value of growth options is given by:

\[
V_O(\theta_H, D) = K_1 D^{\alpha_1} + K_2 D^{\alpha_2}
\]
\[
V_O(\theta_L, D) = K_1 \epsilon_1 D^{\alpha_1} + K_2 \epsilon_2 D^{\alpha_2},
\]
where the constants \(\epsilon_1, \epsilon_2,\) and \(\alpha_1, \alpha_2\) are given in the Appendix.\(^2\)

The optimal option exercise rule is given by a pair of option exercise thresholds, \(\hat{D}(\theta)\) for \(\theta = \theta_H, \theta_L\), such that it is optimal to exercise the growth option in state \(\theta\) if and only if \(D \geq \hat{D}(\theta)\), for \(\theta = \theta_H, \theta_L\). The coefficients, \(K_1\) and \(K_2\), along with the optimal option exercise threshold are jointly determined by the value-matching and smooth-pasting conditions:

\[
\begin{align*}
V_O(\theta, \hat{D}(\theta)) &= a(\theta) \hat{D}(\theta) - 1 \\
\frac{\partial}{\partial D} V_O(\theta, \hat{D}(\theta)) &= a(\theta)
\end{align*}
\]

for \(\theta = \theta_H, \theta_L\) (10)

Proof. See Appendix \(\blacksquare\)

1.2 The Effect of Shocks to Idiosyncratic Volatility

We first consider the economy with time-varying idiosyncratic volatility that corresponds to Equation (4). As we argue in Proposition 2 below, high volatility of idiosyncratic shocks is always associated with a high option value and a delay in option exercise. This result is consistent with the intuition in the standard real option theory. The real option literature typically considers a constant volatility set-up, offering comparative statics for options with different but constant volatilities. Our model incorporates stochastic volatility and allows us to explore the effect of volatility news on option returns.

Proposition 2 In the economy with time-varying idiosyncratic volatility, an increase in

\(^2\)As we show in the Appendix, the general solution of option value is given by \(V_O(\theta_H, D) = \sum_{j=1}^{4} K_j \epsilon_j(\theta) D^{\alpha_j}\), and \(V_O(\theta_L, D) = \sum_{j=1}^{4} K_j D^{\alpha_j}\), where \(\alpha_3 < \alpha_4 < 0\) and \(\alpha_2 > \alpha_1 > 1\). The boundary condition \(V_O(\theta, 0) = 0\) implies that \(K_3 = K_4 = 0\).
**volatility raises option values and the option exercise threshold, that is,**

\[ V_O(\theta_H, D) > V_O(\theta_L, D), \quad \text{for all } D, \]

and

\[ \hat{D}(\theta_H) > \hat{D}(\theta_L). \]

**Proof.** See Appendix. ■

We illustrate the result of the above proposition in Figures 1 and 2. The time-series parameters used in constructing this example are chosen to match the first two moments of annual consumption and dividend growth rates, and our preference configuration implies preference for early resolution of uncertainty. The full list of parameter values is presented in Table 1. The solid line in Figure 1 shows the value of growth options in the high volatility state \((\theta_H)\); the dashed line represents option values in the low volatility state \((\theta_L)\). The option exercise threshold in the low and high volatility states is depicted as a square and a circle, respectively. As the figure shows, the value of growth options is always higher in the high volatility state than in the state when idiosyncratic volatility is low. High volatility is also associated with a high option exercise threshold due to the option value of waiting.

Figure 2 plots exposure of the two types of assets to idiosyncratic volatility risks. The solid line is the ratio of the value of assets in place in two volatility states, \(\frac{V_A(\theta_H, D)}{V_A(\theta_L, D)}\); the dashed line is the corresponding ratio of the value of growth options, \(\frac{V_O(\theta_H, D)}{V_O(\theta_L, D)}\). Notice that the value of growth options always responds positively to volatility news whereas the value of assets in place does not. That is, \(\frac{V_O(\theta_H, D)}{V_O(\theta_L, D)} > 1\) and \(\frac{V_A(\theta_H, D)}{V_A(\theta_L, D)} = 1\). As the figure also shows, sensitivity of option returns to idiosyncratic volatility shocks is higher the further the option is from the option exercise threshold. Thus, asset exposure to variations in idiosyncratic volatility is informative about (i) the type of the asset (option vs. asset in place) and (ii) moneyness of growth options.
1.3 The Effect of Shocks to Aggregate Volatility

In contrast to idiosyncratic volatility, in a general equilibrium setting, aggregate volatility risks affect the stochastic discount factor and, therefore, asset prices. Qualitatively, general equilibrium implications of aggregate volatility for option values depend on preference parameters. We assume that preference parameters satisfy \( \gamma > 1 > 1/\psi \). As shown in Bansal and Yaron (2004), these preferences ensure a rise in discount rates at times of high aggregate uncertainty and are consistent with the empirical evidence of countercyclical dynamics of risk premia.

Consider the economy with time-varying aggregate volatility described in Equation (5). Let \( \hat{D} = \max \{ \hat{D}(\theta_H), \hat{D}(\theta_L) \} \) denote the maximum of the option exercise thresholds. The following proposition characterizes the response growth options to aggregate volatility shocks.

**Proposition 3** Consider the economy with time-varying aggregate volatility and assume that \( \rho = 1. \)\(^3\) Suppose that

\[
\kappa + \beta + \frac{1}{\psi} \mu_C - \frac{1}{2} \gamma \left( 1 + \frac{1}{\psi} \right) \theta^2 > (1 + 2\gamma) \left( \mu_D + \gamma \sigma^2 \right), \quad \text{for} \quad \theta = \theta_H, \theta_L
\]

then

(1) there exists a unique \( D^* \in \left( 0, \hat{D} \right) \) such that

\[
V_O(\theta_H, D) > V_O(\theta_L, D), \quad \text{for all} \quad D \in \left( 0, D^* \right),
\]

and

\[
V_O(\theta_H, D) < V_O(\theta_L, D), \quad \text{for all} \quad D > D^*.
\]

(2) The option exercise thresholds satisfy \( \hat{D}(\theta_H) > \hat{D}(\theta_L) \).

**Proof.** See Appendix ■

\(^3\)The assumption of \( \rho = 1 \) is not critical. We impose it only to simplify the sufficient condition and to facilitate its interpretation.
Fluctuations in aggregate volatility have two effects on option values: the standard volatility (or partial equilibrium) effect and the discount rate (or general equilibrium) effect. While the first effect raises options’ payoffs, the second one causes growth-option values to decline. Depending on moneyness of growth options, one or the other dominates and determines the sign of option exposure to aggregate volatility risks.

Consider first options that are close to the option exercise threshold. Note that under our assumptions on preferences, prices of asset in place are depressed in states of high aggregate uncertainty: $a(\theta_H) < a(\theta_L)$. Intuitively, high aggregate volatility states are associated with high risk premia and, therefore, low price-to-dividend ratios. Hence, at-the-money growth options also respond negatively to aggregate volatility risks: $a(\theta_H) \hat{D} - 1 = V_O(\theta_H, \hat{D}) < V_O(\theta_L, \hat{D}) = a(\theta_L) \hat{D} - 1$. Moreover, given that options are levered positions on assets in place, their values decline more on positive news about aggregate uncertainty compared with assets in place. By continuity, options that are sufficiently close to the option exercise threshold will also feature negative exposure to aggregate volatility. That is, for options that are soon or about to be exercised, discount rate effect dominates and an increase in aggregate volatility leads to a decline in option values.

As options move further away from the option exercise threshold, the partial equilibrium effect becomes more important and, under the condition provided in Proposition 3, eventually dominates the negative discount rate effect. Consider an option that is deep out-of-the-money. An increase in aggregate volatility has a two-fold effect. The general equilibrium effect is still present, pushing the option value down. As the same time, an increase in aggregate volatility raises the probability that the option will eventually end up in the money, pushing its value up. Deep out of the money, the partial equilibrium effect outweighs the discount rate effect. As a result, sufficiently out-of-the-money options have positive exposure to aggregate volatility risks. Note that, in contrast to the case of idiosyncratic volatility, the relationship between option values and aggregate volatility is not uniform: option values increase with volatility when they are deep out-of-the-money but decline when they are close to the option exercise threshold.

Depending on parameter values, the general equilibrium effect may or may not dominate over the entire domain of option moneyness. Proposition 3 provides a sufficient condition
for the discount rate effect to get ultimately overrun by the standard volatility effect.\footnote{As we show in the Appendix, the conclusion of the proposition holds under a much more general condition (provided in Equation (39)). Quantitatively, we find that it holds under a wide range of plausible parameter values.} Intuitively, inequality in Equation (11) requires effective depreciation to be high enough relative to the risk premium. The left hand size of Equation (11) is the sum of the depreciation rate of assets in place and the risk-free interest rate.\footnote{To be precise, $\beta + \frac{1}{\sigma} \mu_C - \frac{1}{2} \gamma \left( 1 + \frac{1}{\sigma} \right) \theta^2$ is the risk-free interest in the economy with constant aggregate volatility $\theta$.} Note that the general equilibrium channel affects the value of growth options through their terminal (on exercise) payoff. Higher risk aversion enhances the general equilibrium effect through an increase in risk premia. But, if the interest rate or depreciation rate are relatively high, then the general equilibrium effect dies off fast enough and the partial equilibrium effect becomes dominant for sufficiently out-of-the-money options. The proposition also shows that, provided Equation (11) holds, the option exercise threshold is governed by the partial equilibrium effect of volatility and it is optimal to delay option exercise in the high volatility state.

We illustrate Proposition 3 in Figures 3 and 4 using the same set of parameter values as in Table 1. Figure 3 plots the value of growth options in the high aggregate volatility state (solid line) and that in the low aggregate volatility state (dashed line). In the region close to the option exercise threshold, the general equilibrium effect dominates and option values are higher when aggregate volatility is low. In contrast, for deep out-of-the-money options, aggregate volatility has a positive effect on option values. In this region, the general equilibrium effect is dominated by the conventional volatility channel due to a higher likelihood that options end up in the money before they disappear.

Figure 4 presents exposure of assets in place and growth options to aggregate volatility risks. Notice that $\frac{\alpha(\theta_H)}{\alpha(\theta_L)} = \frac{V_A(\theta_H, D)}{V_A(\theta_L, D)} < 1$, i.e, the value of assets in place declines on positive volatility news due to an increase in risk premia. The response of growth options, $\frac{V_O(\theta_H, D)}{V_O(\theta_L, D)}$, can be higher or lower than the response of assets in place. When options are considerably out of the money, their values increase in volatility. These options are less risky than assets in place and, in fact, deep out-of-the-money options provide insurance against aggregate volatility shocks. Options that are close to the option exercise threshold are more sensitive to discount rate risks and, therefore, respond more negatively to volatility innovations than
do assets in place.

There are three important implications of our theoretical analysis. First, the value of growth options always increases with idiosyncratic volatility. Hence, price exposure to idiosyncratic volatility news should certainly be informative about differences in growth opportunities across firms. Second, the response of growth options to volatility of aggregate shocks is ambiguous: it can be either positive or negative depending on option moneyness. Third, growth options may have either higher or lower exposure to aggregate volatility news compared with assets in place. The latter two implications suggest that exposure to aggregate volatility is unlikely to reveal differences in option intensity across firms. To summarize, from the theoretical point of view, the amount of growth options owned by firms is best measured by price sensitivity to variation in idiosyncratic volatility. Measures based on aggregate (hence, total) volatility may be contaminated by the discount rate effect and may be powerless to distinguish between growth and asset-in-place intensive firms.

2. Empirical Evidence

Motivated by our theoretical results, we propose to measure growth options by firms’ sensitivity to idiosyncratic volatility news and in this section, we explore its ability to identify option-intensive firms. In particular, we examine if equity exposure to variation in idiosyncratic volatility (which, for brevity, we refer to as “ID-volatility beta”) contains information about firms’ future investment, growth and expected returns. We also test the predictive content of exposure to aggregate volatility (“aggregate volatility beta”) for future investment decisions of firms.

2.1 Firm Data

All the cross-sectional data come from Compustat and the Center for Research in Securities Prices (CRSP). We focus on non-financial firms whose common shares are traded on NYSE, AMEX and Nasdaq. We collect return and price series, the number of outstanding shares, capital expenditure (to measure investment), property, plant and equipment (to measure the
amount of capital), expenditure on research and development (R&D), book value of assets, sales and cash holdings. For each firm in the sample, we compute its book-to-market ratio as in Fama and French (1993), its leverage as a ratio of short- and long-term debt to the sum of debt and market value of equity, and Tobin’s Q as a ratio of the sum of market capitalization, book value of preferred equity and long-term debt less inventories and deferred taxes to the sum of book value of common and preferred equity and long-term debt. We use data sampled on daily, monthly, and annual frequency. Monthly and annual data are converted to real using the consumer price index from the Bureau of Labor Statistics. The overall coverage of the data is from 1964 to 2012.

2.2 Aggregate and Idiosyncratic Volatility Measures

We measure aggregate and firm-level volatility by realized variance of equity returns.\(^6\) Monthly series of aggregate variance are constructed by summing up squared daily returns of the aggregate market portfolio. Since our focus is on volatility news and their effect on prices, we work with innovations rather than levels. Aggregate volatility news are extracted by applying an AR(1) filter to the logarithm of the market variance.\(^7\) We choose to measure aggregate volatility using market equity rather than consumption data since the latter are not available at high frequencies.

Idiosyncratic volatility is constructed in two steps. First, we estimate firm-level volatility as in Campbell, Lettau, Malkiel, and Xu (2001), and Brandt, Brav, Graham, and Kumar (2010) using industry-adjusted daily returns. In particular, for firm \(i\) that belongs to industry \(J\), its variance in month \(t\) is measured as:

\[
V_{i,t}^{FL} = \sum_{d \in t} \left( R_{i,d} - \bar{R}_{J,d} \right)^2,
\]

where \(R_{i,d}\) and \(\bar{R}_{J,d}\) are daily returns of firm \(i\) and industry \(J\), respectively.\(^8\) We find that the

\(^6\)We use terms “volatility” and “variance” interchangeably.

\(^7\)Strong time-dependence in volatility series has been well recognized in the literature (eg., Bollerslev, Chou, and Kroner (1992), and Andersen, Bollerslev, Diebold, and Ebens (2001)).

\(^8\)We use the 30-industry classification available at Kenneth French’s data library. Allowing for non-unit industry betas does not affect our empirical evidence. We provide a more detailed discussion of the robustness of our findings below.
average correlation between firm-level and aggregate volatility is fairly high, of about 28%.
Thus, the industry adjustment alone is not sufficient to remove all systematic variation.
To further isolate purely idiosyncratic movements in volatility, we orthogonalize firm-level variance with respect to aggregate variation. Specifically, innovation in idiosyncratic volatility of firm \( i \) in month \( t \) (denoted by \( \sigma_{i,t}^{ID} \)) is measured by the residual in the following regression:

\[
v_{i,t}^{FL} = k_{i,0} + k_{i,1} v_t^M + k_{i,2} v_{i,t-1}^{FL} + \sigma_{i,t}^{ID}, \tag{13}
\]

where \( v_{i,t}^{FL} \equiv \log \left( V_{i,t}^{FL} \right) \), and \( v_t^M \) is the logarithm of the aggregate market variance. An autoregressive term is included to filter out any remaining persistence. Note that our estimation procedure aims to identify time-variation in firm-specific volatility without taking a strong stand on the asset pricing model that governs equity returns.

### 2.3 Exposure to Idiosyncratic Volatility

To explore the predictive ability of ID-volatility betas for future investment, we sort firms on their exposure to idiosyncratic volatility and compare growth-related characteristics of the resulting portfolios. We measure idiosyncratic volatility exposure, which we denote by \( \beta^{ID} \), using 3-year rolling window regressions. In particular, at the end of a given year, we regress the logarithm of firm returns on innovations in its idiosyncratic volatility using monthly data over the previous three years. We then sort firms on their ID-volatility exposure into five value-weighted portfolios and hold them for one year. Next December, we re-estimate volatility betas by rolling the estimation window one year forward, and repeat the sorting procedure. Table 2 provides a description of the sorted portfolios. We show two sets of statistics: characteristics at the time when portfolios are formed and portfolio statistics over the holding period.

Consistent with the theoretical prediction, we find that equity returns have mostly positive exposure to news in idiosyncratic volatility. That is, equity prices tend to increase in response to a rise in idiosyncratic volatility. As we argued earlier, a positive response is likely to be driven by growth options and its magnitude depends on the amount and moneyness of options available to firms. We find that firms with high exposure to idiosyncratic volatility are
characterized by high Tobin’s Q, high cash holdings, low amount of capital and low leverage. High-β ID firms account for only 4% of the total capital while low-β ID firms contribute a much larger share, of about 31%, to the total capital stock. Firms in the top portfolio, on average, have a 12% ratio of cash to assets and a 20% leverage. The corresponding statistics of firms in the bottom portfolio are 7% and 25%, respectively. In addition, firms with high exposure to idiosyncratic volatility tend to spend more on research and development compared with the others. The ratio of R&D expense to lagged assets of the top quintile portfolio is 5.5%, which is about 2% larger than that of the rest of the market.

As further shown in Table 2, forward-looking characteristics vary substantially across portfolios sorted on idiosyncratic volatility exposure. Firms with high β ID feature higher growth in sales and investment in the year following the portfolio formation compared with low-exposure firms. As β ID increases, the average growth in sales increases monotonically from 2.6% to 4.7%. The difference in investment growth rates is more striking: the average growth of investment changes from -0.7% to 8.1% from the bottom to the top quintile portfolios.

We also find that high-β ID firms are characterized by high prices, low dividends and low expected returns. The sample mean of the price-dividend ratio of the top quintile portfolio is almost 159 while it is only 42 for the bottom quintile. As idiosyncratic volatility exposure goes up, the average return declines from about 7.5% to 5.3%. This evidence should not be interpreted as a puzzle. Firm-specific volatility is purely idiosyncratic. Hence, exposure to innovations in idiosyncratic volatility should not be priced. However, if as we argue ID volatility exposure provides a signal about relative composition of firms’ assets, and growth options and assets in place have different risk characteristics, then sorting on idiosyncratic volatility beta would reveal differences in systematic risks and risk premia of growth and value assets. Our findings suggest that growth-option intensive firms are less risky and, hence, carry smaller premia compared with asset-in-place intensive firms, which is consistent with theoretical predictions in Ai and Kiku (2013), Kogan and Papanikolaou (2012), and Ai, Croce, and Li (2013).
2.4 Controlling for Book-to-Market

The empirical evidence presented in Table 2 shows that, unconditionally, idiosyncratic volatility exposure is able to predict firms’ future investment and growth. The one-dimensional sort, however, cannot reveal if idiosyncratic volatility betas provide additional information beyond of what we can learn from the conventional, book-to-market based classification of growth and value. To address this issue, we consider a double sort on book-to-market and $\beta^{ID}$. We first sort all firms into three book-to-market portfolios, then we split each BM bin into three portfolios with low, medium and high ID-volatility beta. Portfolios are value weighted and rebalanced on an annual frequency.$^9$ Table 3 presents some of the key characteristics of the resulting portfolios. To conserve space, we report statistics only for corner portfolios, those at the intersection of low and high book-to-market and low and high ID-volatility beta.

The evidence in Table 3 confirms the well-known ability of book-to-market ratio to identify option-intensive firms. Portfolios with low ratio of book to market feature significantly higher R&D expenditure and invest at a much higher rate relative to firms with high book-to-market characteristic. In fact, while firms in the low book-to-market portfolio have strongly positive investment growth rates, high-BM firms undergo a decline in investment, on average.

Importantly, we find that after controlling for book-to-market characteristic, exposure to idiosyncratic volatility helps further separate out high and low expected growth firms, especially across firms with low BM ratio. The average sales growth of low-BM and low-$\beta^{ID}$ firms is about 6.05%. Keeping BM ratio fixed, the growth rate increases to 11.1% for firms with high idiosyncratic volatility exposure. The increase in investment growth is more pronounced. The average investment growth of firms with low book-to-market ratio more than triples from 4.9% to 16.2% as firms’ exposure to idiosyncratic volatility changes from low to high.

Note that firms with low book-to-market ratios are typically firms with relatively large market capitalization. Hence, the evidence of a significant $\beta^{ID}$-effect among firms with low

---

$^9$Empirical evidence based on an independent two-dimensional sort is very similar to the one presented here.
BM ratios suggests that it is not driven by very small firms. Indeed, controlling for size, we find that the effect is especially strong amongst medium and large firms. In particular, across medium-sized firms, those with low ID-volatility exposure have on average 6.3% decline in investment, while those with high exposure to idiosyncratic volatility tend to increase investment by 6%.

### 2.5 Idiosyncratic Volatility Exposure and Future Investment

To formally quantify the extent to which idiosyncratic volatility exposure is able to account for unobservable growth opportunities, we test its ability to forecast firms’ future investment. We consider two specifications. In the first specification, we use ID-volatility betas to run the following predictive regression:

\[
\log \frac{I_{i,t+k}}{I_{i,t}} = \phi_0 + \phi \beta_{ID}^{i,t} + \varphi X_{i,t} + u_{i,t+1}, \tag{14}
\]

where the left-hand side variable is the logarithm of cumulative (annualized) investment growth of firm \(i\) (i.e., \(I_{i,t+k} \equiv \frac{1}{k} \sum_{j=1}^{k} I_{t+j}\)), \(\beta_{ID}^{i,t}\) is firm-\(i\) exposure to idiosyncratic volatility at time \(t\), and \(X_{i,t}\) is a vector of controls. Our focus here is on the magnitude and significance of the slope coefficient \(\phi\).

In our second specification, instead of using volatility exposure directly, we use dummy variables that represent the location of each firm within ID-volatility beta-sorted portfolios. That is, we estimate:

\[
\log \frac{I_{i,t+k}}{I_{i,t}} = \phi_0 + \sum_{p=2}^{5} \phi_j D_{i,t}^{(p)} + \varphi X_{i,t} + u_{i,t+1}, \tag{15}
\]

where \(D_{i,t}^{(p)}\) is a dummy variable that equals one if firm \(i\) belongs to portfolio \(p\) at time \(t\), and all other variables are defined as in Equation (14). One potential advantage of the second specification is that might help reduce firm-specific noise coming form the estimated betas.

We consider several variations of each specification: with and without firm fixed effects, and with and without controls. In regression specifications with control variables we include
firm characteristics that are known to predict future investment. We use ratios of sales to assets, cash to capital, book to market and investment to capital, as well as firm market share, return and Tobin’s Q. Predictability of the one-year ahead investment growth (i.e., \( k = 1 \)) is presented in Table 4. The four columns (“Model I” through “Model IV”) correspond to different regression specifications. Panel A reports the estimate of \( \phi \) in Equation (14), Panel B shows the estimates of \( \phi_j \) in specification (15), t-statistics are reported in parentheses. To ensure robustness of our inference to both cross-sectional dependence in errors and residual correlation across time, we cluster standard errors by firm and time.

The first two columns show that exposure to idiosyncratic volatility, by itself, is a strongly significant predictor of future investment growth. Firms with higher ID-volatility betas invest at a much higher rate than do firms that are less sensitive to idiosyncratic volatility news. This evidence is consistent across the two specifications in Equations (14) and (15), and is robust to the inclusion of firm fixed effects. Controlling for firm characteristics makes the magnitude of the \( \beta_{ID} \)-effect decline, and yet it remains strongly significant. The estimates of \( \phi \) in the last two columns are all positive, and the estimates on portfolio dummies, \( \hat{\phi}_j \), are monotonically increasing across quintiles. Quantitatively, controlling for firm characteristics and fixed effects, a one standard deviation increase in idiosyncratic volatility exposure results in a significant 4% increase in investment growth. Put differently, firms in the top \( \beta_{ID} \)-beta portfolio invest by 10% more than do otherwise identical firms in the bottom portfolio. This evidence reveals that firms’ exposure to idiosyncratic volatility contains significant independent information about firms’ future investment decisions (hence, available growth options) over and beyond of what is already captured by such powerful predictors as cash balances, book-to-market ratio, past investment and returns.

Earlier, Grullon, Lyandres, and Zhdanov (2012) find that sensitivity of stock returns to total firm volatility is correlated with conventional measures of growth options. Unlike them, we discriminate between idiosyncratic and aggregate components of volatility, and show that exposure to idiosyncratic volatility is not simply related to the traditional proxies but provides independent information about firms’ future investment. We also highlight the importance of separating idiosyncratic and aggregate volatility movements for both understanding price reaction to different volatility news and for accurate measurement of
growth options.

As discussed earlier, price sensitivity to idiosyncratic volatility captures not only the relative amount of growth options but also their moneyness. Therefore, we expect the effect of ID-volatility exposure on future investment to be persistent as options that are sufficiently out of the money may not be able to reach the option exercise threshold that soon and may spur investment only several years later. We test this hypothesis in Table 5. It shows the response of one-, two- and three-year ahead cumulative investment growth and investment rate to ID-volatility betas, controlling for firm fixed effects and firm characteristics. Investment-to-capital ratio of firm $i$ for horizon $k$ is defined as $\log \frac{\bar{I}_{i,t+k}}{K_{i,t}}$, where $\bar{I}_{i,t+k}$ is the cumulative (annualized) investment and $K_{i,t}$ is the book value of firm capital at time $t$. Consistent with our hypothesis, we find that firms with higher sensitivity to idiosyncratic volatility feature higher rates of investment relative to firms with low exposure even three years out. The magnitude of the estimated slope tends to decline with the horizon but it remains statistically significant. For example, for investment rate, the estimated coefficients are 0.61, 0.51 and 0.48 for horizons of one-, two and three years, respectively, with the corresponding t-statistics of 5.76, 4.33, and 4.55.

The last set of columns in Table 5 characterizes the effect of idiosyncratic volatility exposure on future firm returns. The evidence is based on the regression specification (14) where the left-hand side variable is replaced by the cumulative (annualized) return. We narrow a set of controls to firm characteristics that are known to capture variation in expected returns and remain statistically significant. In particular, we control for sales-to-assets and book-to-market ratios, investment rate, market capitalization, and include firm fixed effects. We find that idiosyncratic volatility exposure has a negative effect on expected returns. Quantitatively, a one standard deviation increase in ID-volatility beta lowers next-year return by about 2.2%. The effect is significant at the 10% level at the one- and two-year horizons, and at the 5% level at the three-year horizon. This evidence is consistent with the well-known value premium. In the data, growth firms identified by low book-to-market ratio on average have lower rates of returns compared to value or high book-to-market firms. We identify growth firms by their high exposure to idiosyncratic volatility and also find that they carry low expected returns even after controlling for book-to-market and size.
2.6 Robustness

To confirm our findings we conduct a series of robustness checks. First, we construct idiosyncratic volatility by adjusting returns using the Fama and French (1993) three-factor model instead of industry portfolios. Second, we use our benchmark volatility series but vary the length of the estimation window over which ID-volatility exposure is measured. We consider two alternatives: two- and five-year windows that are not too short or too long. Table 6 presents average growth rates in sales and investment of quintile portfolios ranked on alternative ID-volatility betas. For brevity, we report only two characteristics as other cross-sectional patterns are both qualitatively and quantitatively consistent with our benchmark findings. As the table shows, our empirical evidence is robust – across the alternative measures, exposure to idiosyncratic volatility is highly informative about future investment and growth. Sorting on alternative ID-volatility betas results in a monotonic increase in future sales growth and generates a sizable spread in average investment growth rates between high- and low-exposure firms. For example, with the Fama-French adjustment, the difference in investment growth between the top and the bottom portfolios exceeds 10%. In regression settings, alternative ID-volatility exposure remains a significant predictor of firms’ future investment after controlling for firm characteristics. To conserve space this evidence is not presented and is available upon request.

We focus on quintile portfolios to ensure that they are not too thin. In a decile sort, the cross-sectional dispersion in investment growth and returns is further amplified. In particular, the difference in average sales growth between the top and the bottom decile portfolios is 3.2% (5.5% v.s. 2.3%), the spread in average investment growth is 13% (10.2% v.s. -2.8%), and the dispersion in average returns is 6.2% (3.5% of the high ID-volatility beta portfolio v.s. 9.7% of the bottom decile).10

10These moments are based on our benchmark measure of $\beta^{ID}$. Alternative measures produce similar evidence.
2.7 Idiosyncratic Volatility: Exposure v.s. Level

Idiosyncratic volatility has recently attracted a lot of attention in the asset pricing literature. Campbell, Lettau, Malkiel, and Xu (2001), Cao, Simin, and Zhao (2008), Brandt, Brav, Graham, and Kumar (2010) focus on understanding time-series dynamics of aggregate idiosyncratic volatility. Ang, Hodrick, Xing, and Zhang (2006, 2009) analyze the cross-sectional relationship between the level of idiosyncratic volatility and average returns. Our paper aims to understand if price sensitivity to idiosyncratic volatility news reveals information about growth options owned by firms.

Ang, Hodrick, Xing, and Zhang (2006, 2009) show that firms with high level of idiosyncratic volatility have puzzlingly low average returns. As we show, firms with high exposure to idiosyncratic volatility also carry low premia and, in addition, are expected to invest at a high rate. One might think that high level of idiosyncratic volatility and high exposure to firm-specific volatility are highly correlated and that our findings are just a restatement of the idiosyncratic volatility puzzle. We argue that it is not the case. Table 7 provides evidence in support of our argument. It shows average returns and investment growth rates of $3 \times 3$ portfolios constructed by double-sorting firms according to the level of their idiosyncratic volatility and ID-volatility exposure. We present evidence based on two sorts. In Panel A, we first rank firms on the level of idiosyncratic volatility, and then divide the resulting portfolios into three $\beta_{ID}$ bins. In Panel B, we present the intersections of portfolios sorted independently on each characteristic.\footnote{In Table 7, we continue to rely on our benchmark measure of idiosyncratic volatility and exposure. The evidence based on alternative measures discussed in the robustness section above is very similar.}

Is high idiosyncratic volatility informative about future investment? Sometimes but not always. For firms with relatively high ID-volatility betas, average investment growth overall increases with the level of firm-specific volatility. However, among firms with low sensitivity to idiosyncratic volatility, the pattern in investment growth is completely the opposite – in this cohort, highly volatile firms tend to cut down their future investment. For example, based on independent sort, investment growth of firms with high $\beta_{ID}$ increases from 5.7% to 8.6% when idiosyncratic volatility changes from low to high but across firms with low $\beta_{ID}$ it declines from 0.7% to -6.3%, respectively. In contrast, high exposure to ID-
volatility consistently signals high investment growth for any level of idiosyncratic volatility. Controlling for the level of firm-specific volatility, average investment growth monotonically increases in $\beta^{ID}$ in both sorts. The spread in average growth rates between high and low ID-volatility beta portfolios varies between 5% and 15%.

Similar to our earlier evidence, expected returns tend to decline in ID-volatility exposure (except for firms with low level of idiosyncratic volatility that feature no discernible dispersion in mean returns). Under the two sorting schemes, as $\beta^{ID}$ changes from low to high, the average return declines by about 1.4% and 3.8% for firms with median and high level of idiosyncratic volatility, respectively. Consistent with idiosyncratic volatility puzzle literature, we find that firms with high level of idiosyncratic volatility have low rates of return. What is interesting is that this result appears to be strong only among firms with high $\beta^{ID}$. For them, the average return declines by about 4.3%, on average, as the level of firm-specific volatility changes from low to high. For firms with low and median ID-volatility exposure, the corresponding spread is only 0.5%.

2.8 Exposure to Aggregate Volatility and Future Growth

As we showed both theoretically and empirically exposure to idiosyncratic volatility is informative about cross-sectional differences in growth options. In this section we examine if price sensitivity to aggregate volatility risks is also able to reveal differences in growth characteristics across firms. Table 8 presents summary statistics of portfolios sorted on exposure to news about aggregate volatility. Aggregate volatility exposure, which we denote by $\beta^{A}$, is estimated similarly to ID-volatility betas by running three-year rolling window regressions of log equity returns on innovations in aggregate volatility.

We find exposure to aggregate volatility to be largely negative. That is, equity prices tend to fall during times of high economic uncertainty, which is consistent with price reaction of portfolio-level returns documented in Bansal, Khatchatrian, and Yaron (2005), and Bansal, Kiku, Shaliastovich, and Yaron (2013). In our sample, four out of five firms experience low returns when market volatility goes up. This is a manifestation of the discount rate effect – high aggregate uncertainty rises risk premia and lowers asset prices. Aggregate volatility risks
carry a negative price. Firm with high exposure to aggregate uncertainty (i.e., firms with highly negative $\beta^A$) earn high risk premia. The average return is monotonically declining in $\beta^A$, with firms in the bottom quintile earning about 9% and firms in the top quintile earning only 5% per annum.

While we find a strong and robust dispersion in risk premia, we observe no discernible pattern in investment- and growth-related characteristics across portfolios sorted on exposure to aggregate volatility. Along some characteristics, such as Tobin’s Q, leverage and R&D spending, firms with low aggregate volatility betas appear marginally more option-intensive relative to firms with high betas. However, the average investment growth of the top quintile portfolio turns out to be higher than that of the bottom portfolio (2.6% vs. 0.7%), and the correlation between aggregate volatility betas and future growth in sales is virtually zero. We also fail to find any conclusive variation in future investment and sales across portfolios sorted on $\beta^A$ after we control for either book-to-market or size.

Table 9 shows more rigorously that sensitivity of equity prices to aggregate volatility news does not help predict future firm-level investment in a significant way. The table presents the estimates of the slope coefficients in panel regressions of one-year ahead investment growth on aggregate volatility beta. We exploit the same specifications as in Equations (14) and (15) replacing ID-volatility betas with $\beta^A$s. As the table shows, the ID-$\beta^A$ effect is positive but lacks significance even in firm controls are excluded.

To further highlight differences between exposure to aggregate and idiosyncratic volatility news, we consider a double sort on the two betas. Table 10 presents characteristics of $3 \times 3$ portfolios constructed by ranking firms independently on $\beta^A$ and $\beta^{ID}$. As the table shows, cross-sectional differences in investment characteristics are primarily driven by differences in exposure to idiosyncratic volatility. Controlling for $\beta^A$, average R&D spending, sales and investment growth increase by about 1%, 2%, 8%, respectively, from low- to high-$\beta^{ID}$ bins. The dispersion across $\beta^A$-terciles is conflicting and, if anything, is quite weak. To summarize, we find no convincing evidence that exposure to aggregate volatility helps differentiate between option- and asset-in-place-intensive firms.
3. Conclusion

We argue, both theoretically and empirically, that price exposure to idiosyncratic volatility news is informative about cross-sectional differences in growth opportunities. We introduce two types of volatility risks in a standard real-option setting: risks due to variation in volatility of idiosyncratic shocks, and risks due to variation in volatility of aggregate shocks. Because variation in idiosyncratic volatility has no effect on marginal utility, growth-option values always rise on positive news in idiosyncratic volatility. In contrast, due to the discount-rate effect, the response of growth options to aggregate volatility news is ambiguous. Under fairly general and empirically plausible conditions, an increase in aggregate volatility raises discount rates and lowers valuations. Hence, the discount-rate and the conventional volatility channels work against each other. We show that, in general, the discount-rate effect dominates for options that are close to the option exercise threshold, and the traditional volatility effect dominates when options are deep out of the money. We also show that the response of growth options to aggregate volatility news may exceed or be lower than the corresponding exposure of assets in place.

Guided by our theoretical results, we propose to measure growth-option intensity by firms’ exposure to idiosyncratic volatility news. We show that in the data, equity exposure to innovations in idiosyncratic volatility is a significant predictor of firms’ future investment and growth. Importantly, information carried by idiosyncratic volatility exposure is not subsumed by the conventional measures of growth such as book-to-market ratio or Tobin’s Q, and helps identify option-intensive firms even after controlling for relevant firm characteristics. We also show that firms with high exposure to idiosyncratic volatility, on average, have lower premia compared with firms that show low response. This evidence suggests that growth options are less risky than assets in place and confirms the well-known differences in risk premia in the cross-section of book-to-market sorted portfolios. Further, we find no significant evidence that exposure to aggregate volatility is informative about cross-sectional differences in the relative composition of firms’ assets. In all, our paper emphasizes the importance of distinguishing between idiosyncratic and aggregate volatility shocks in understanding firms’ growth opportunities.
References


4. Appendix

4.1 Proof of Proposition 1

We first derive the pricing kernel of the economies with aggregate and idiosyncratic volatility shocks. The representative agent’s intertemporal preference is represented by the Kreps and Porteus (1978) utility with constant relative risk aversion parameter $\gamma$ and constant intertemporal elasticity of substitution parameter $\psi$. We use $\{V_t\}_{t \geq 0}$ to denote the utility process of the representative agent. The representative consumer’s preference is specified by a pair of aggregators $(f, A)$ such that:

$$dV_t = [−f(C_t, V_t) − \frac{1}{2} A(V_t)||\sigma_V(t)||^2]dt + \sigma_V(t)dB_t$$ (16)

We adopt the convenient normalization $A(v) = 0$ ((Duffie and Epstein 1992b)), and denote $\bar{f}$ the normalized aggregator. In this case, $\bar{f}(C, V)$ is:

$$\bar{f}(C, V) = \frac{\beta}{1 - \frac{1}{\psi}} C^{1-1/\psi} - \left(\frac{1 - \gamma}{V} \frac{1}{\frac{1}{\psi} \frac{1}{1-\gamma}}\right).$$ (17)

Due to the homogeneity of the utility function, it can be represented as

$$V(\theta, C) = \frac{1}{1 - \gamma} H(\theta) C^{1-\gamma}$$ (18)

for some function $H(\theta)$. Under Assumption (6), $H(\theta)$ is given by:

$$H(\theta_H) = \left\{\frac{1}{\beta} \left[\beta + \frac{1}{2} \gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2(\theta_H) - \left(1 - \frac{1}{\psi}\right) \mu_C - \frac{1 - \frac{1}{\psi}}{1 - \gamma} \lambda (\omega^{-1} - 1)\right]\right\}^{-\frac{1}{1-\frac{1}{\theta_H}}}$$

$$H(\theta_L) = \left\{\frac{1}{\beta} \left[\beta + \frac{1}{2} \gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2(\theta_L) - \left(1 - \frac{1}{\psi}\right) \mu_C - \frac{1 - \frac{1}{\psi}}{1 - \gamma} \lambda (\omega - 1)\right]\right\}^{-\frac{1}{1-\frac{1}{\theta_L}}}$$

where $\omega \geq 1$ is the unique solution to the following equation on $(0, \infty)$:

$$\omega^{-\frac{1}{1-\frac{1}{\theta_H}}} = \frac{\beta + \frac{1}{2} \gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2(\theta_H) - \left(1 - \frac{1}{\psi}\right) \mu_C - \frac{1 - \frac{1}{\psi}}{1 - \gamma} \lambda (\omega^{-1} - 1)}{\beta + \frac{1}{2} \gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2(\theta_L) - \left(1 - \frac{1}{\psi}\right) \mu_C - \frac{1 - \frac{1}{\psi}}{1 - \gamma} \lambda (\omega - 1)}.$$ (19)

The above results include both the economy with idiosyncratic volatility shocks and the economy with aggregate volatility shocks as special cases. In the economy with idiosyncratic volatility shocks,
\( \sigma_C (\theta_H) = \sigma_C (\theta_L) = \sigma \) and \( \omega = 1 \). In this case, \( H (\theta_H) = H (\theta_L) \). In the economy with aggregate volatility shocks, \( \sigma_C (\theta_H) > \sigma_C (\theta_L) \) and \( \omega > 1 \). Given our assumption of \( \gamma > 1 \), this implies that high aggregate volatility is associated with lower utility of the representative agent.

Let \( \{\pi_t\}_{t=0}^{\infty} \) denote the marginal utility process of the representative agent. To derive an explicit expression of the law of motion of \( \pi_t \), it is convenient to represent the Markov process \( \{\theta_t\}_{t=0}^{\infty} \) as integration with respect to counting processes:

\[
d\theta_t = \Delta \eta (\theta_t^-)^T dN_t, \tag{20}
\]

where
\[\Delta = \theta_H - \theta_L,\]

and
\[N (t) = [N_{Ht}, N_{Lt}]^T\]

are independent counting processes with intensity \( \lambda \) (For simplicity, we assume that \( N_{Ht} \) and \( N_{Lt} \) have the same intensity parameter, \( \lambda \)). The function \( \eta (\theta) \) in equation (20) is given by:

\[
\eta (\theta) = \begin{bmatrix} -I_{\{\theta_H\}} (\theta) , I_{\{\theta_L\}} (\theta) \end{bmatrix},
\]

where \( I_x \) is the indicator function:

\[
I_{\{x\}} (y) = \begin{cases} 
1 & \text{if } y = x, \\
0 & \text{if } y \neq x.
\end{cases}
\]

Using the results in (Duffie and Epstein 1992b)),

\[
\frac{d\pi_t}{\pi_t} = \frac{d\bar{f}_C (C_t, V_t)}{\bar{f}_C (C_t, V_t)} + \bar{f}_V (C_t, V_t) dt. \tag{21}
\]

Using equation (18) and (21), one can show that

\[
d\pi_t = \pi_t \left[ -\bar{\bar{r}} (\theta_t) dt - \gamma \sigma_C (\theta) dB_t - \eta_{\pi} (\theta_t^-)^T dN_t \right],
\]

30
where

\[
\bar{r} (\theta_L) = \beta + \frac{1}{\psi} \mu_C - \frac{1}{2} \gamma \left( 1 + \frac{1}{\psi} \right) \sigma_C^2 (\theta_L) + \frac{1}{\psi} \frac{1 - \gamma}{1 - \gamma} \lambda (\omega - 1),
\]

(22)

\[
\bar{r} (\theta_H) = \beta + \frac{1}{\psi} \mu_C - \frac{1}{2} \gamma \left( 1 + \frac{1}{\psi} \right) \sigma_C^2 (\theta_H) + \frac{1}{\psi} \frac{1 - \gamma}{1 - \gamma} \lambda (\omega^{-1} - 1),
\]

(23)

and

\[
\eta (\theta) = \left[ (1 - \hat{\omega}^{-1}) I_{\{H\}} (\theta), (1 - \hat{\omega}) I_{\{L\}} (\theta) \right],
\]

with \( \hat{\omega} = \omega \frac{1 / \psi - \gamma}{1 - \gamma} \).

The value of assets in place is given by the following lemma.

**Lemma 1** The value of assets in place is linear in dividend as in (8), where

\[
a (\theta_L) = \frac{\beta_H - \mu_D + \lambda (1 + \hat{\omega}^{-1})}{(\beta_H - \mu_D + \lambda) (\beta_L - \mu_D + \lambda) - \lambda^2},
\]

(24)

\[
a (\theta_H) = \frac{\beta_L - \mu_D + \lambda (1 + \hat{\omega})}{(\beta_H - \mu_D + \lambda) (\beta_L - \mu_D + \lambda) - \lambda^2}.
\]

(25)

In addition, \( a (\theta_L) = a (\theta_H) \) in the case of idiosyncratic volatility shock. Under the assumption \( \gamma > 1 > \frac{1}{\psi} \), \( a (\theta_L) > a (\theta_H) \) in the economy with aggregate volatility shocks.

**Proof.** Consider an asset with the cash flow process \( \{ CF (\theta_t, D_t) \}_{t=0}^{\infty} \), where cash flow is a function of \( (\theta_t, D_t) \). The value function is given by:

\[
V (\theta, D) = E \left[ \int_{0}^{\tau} \frac{\pi_t}{\pi_0} CF (\theta_t, D_t) dt + \frac{\pi_\tau}{\pi_0} B (\theta_\tau, D_\tau) \right] \theta_0 = \theta, D_0 = D.
\]

(26)

Standard result implies that

\[
\pi_t CF (\theta_t, D_t) + \mathcal{L} V (\theta_t, D_t) = 0,
\]

(27)

where

\[
\mathcal{L} V (\theta_t, D_t) = \lim_{\Delta \to 0} \frac{1}{\Delta} E_t [V (\theta_{t+\Delta}, D_{t+\Delta}) - V (\theta_t, D_t)].
\]
Using generalized Ito’s formula, we have:

\[
\bar{r} (\theta) V (\theta, D) = CF (\theta, D) + \frac{\partial}{\partial D} V (\theta, D) D \left[ \mu_D - \gamma \sigma_C^2 (\theta) \right] \\
+ \frac{1}{2} \frac{\partial^2}{\partial D^2} V (\theta, D) \rho^2 \left[ \sigma_C^2 (\theta) + \sigma_D^2 (\theta) \right] \\
+ I_{\{\theta_H\}} (\theta) \lambda \left[ \hat{\omega}^{-1} V (\theta_L, D) - V (\theta_H, D) \right] \\
+ I_{\{\theta_L\}} (\theta) \lambda \left[ \hat{\omega} V (\theta_H, D) - V (\theta_L, D) \right].
\]

(28)

The valuation of assets in place is a special case where \( CF (\theta, D) = D \) and \( \tau = \infty \). In this case, \( V (\theta, D) \) is linear and (28) implies that \( a (\theta) \) is given by (24) and (25).

Denote

\[
\tau^2_L = \rho^2 \left[ \sigma_C^2 (\theta_L) + \sigma_D^2 (\theta_L) \right] \\
\tau^2_H = \rho^2 \left[ \sigma_C^2 (\theta_H) + \sigma_D^2 (\theta_H) \right],
\]

(29)

(30)

\[
\mu_L = \mu_D - \gamma \rho \sigma_C^2 (\theta_L) \\
\mu_H = \mu_D - \gamma \rho \sigma_C^2 (\theta_H),
\]

(31)

(32)

and

\[
r_L = \bar{r} (\theta_L) + \kappa + \lambda \\
r_H = \bar{r} (\theta_H) + \kappa + \lambda
\]

Let \( f_L (\alpha) \) and \( f_H (\alpha) \) be quadratic functions of \( \alpha \) given by:

\[
f_L (\alpha) = \frac{1}{2} \tau^2_L \alpha^2 + \left( \mu_L - \frac{1}{2} \tau^2_L \right) \alpha - r_L, \\
f_H (\alpha) = \frac{1}{2} \tau^2_H \alpha^2 + \left( \mu_H - \frac{1}{2} \tau^2_H \right) \alpha - r_H.
\]

The following lemma characterizes the solution to the quartic equation

\[
f_H (\alpha) f_L (\alpha) - \lambda^2 = 0.
\]

(33)

**Lemma 2** Equation (33) has two strictly positive roots that satisfies \( 1 < \alpha_1 < \alpha_2 \).
Proof. Note that under the assumptions of the parameter values, the quadratic equation $f_L(\alpha) = 0$ has two roots, $\alpha^-_L$ and $\alpha^+_L$ that satisfy $\alpha^-_L < 0 < 1 < \alpha^+_L$. Similarly, $f_H(\alpha) = 0$ has two roots that satisfy $\alpha^-_H < 0 < 1 < \alpha^+_H$. Note also, $f_H(1) f_L(1) - \lambda^2 = (r_L - \mu_L)(r_H - \mu_H) - \lambda^2 > 0$ and $f_H(\alpha) f_L(\alpha) - \lambda^2 > 0$ for $\alpha$ large enough. Therefore, we have $f_H(1) f_L(1) - \lambda^2 > 0$, $f_H(\alpha) f_L(\alpha) - \lambda^2 > 0$, for $\alpha$ large. Note also that $f_H(\alpha^+_L) f_L(\alpha^+_L) - \lambda^2 = f_H(\alpha^+_H) f_L(\alpha^+_H) - \lambda^2 = -\lambda^2 < 0$. This means that equation (33) must have a root on $(1, \min\{\alpha^+_L, \alpha^+_H\})$ and another root on $(\max\{\alpha^+_L, \alpha^+_H\}, \infty)$. Furthermore, it can be verified that $\alpha_1$ and $\alpha_2$ are the only two positive roots of (33).

For $i = 1, 2$, we define

$$e(\alpha_i) = \frac{-f_H(\alpha_i)}{\lambda \omega} = \frac{\lambda \omega^{-1}}{-f_L(\alpha_i)},$$

where the second equality is because $\alpha_1$ and $\alpha_2$ are solutions to (33). Note that $\alpha_2 > \max\{\alpha^+_L, \alpha^+_H\}$. Hence $f_H(\alpha_2), f_L(\alpha_2) > 0$. It follows from the definition of $e(\alpha)$ in equation (34) that $e(\alpha_2) < 0$. Similarly, $\alpha_1 < \min\{\alpha^-_L, \alpha^-_H\}$ and hence $f_H(\alpha_1) < 0$ and $e(\alpha_1) > 0$. Using these notations, the value of growth options are given by the following lemma:

**Lemma 3** Let $e_1 = e(\alpha_1)$ and $e_2 = e(\alpha_2)$, then the value of growth options are given by the equation (9) in proposition 1, where $K_1, K_2$ and the option exercise thresholds, $\hat{D}(\theta_H)$ and $\hat{D}(\theta_L)$ are determined by the value matching and smooth pasting conditions, (10).

Proof. The valuation of options can be viewed as a special case of (26) where $CF(\theta, D) = 0$, $B(\theta_r, D_r) = V(\theta_r, D_r) - 1$ and the stopping time $\tau$ takes the following form:

$$\tau = \inf \left\{ t : D_t > \hat{D}(\theta_H), \theta_t = \theta_H \right\} \cap \left\{ t : D_t > \hat{D}(\theta_L), \theta_t = \theta_L \right\},$$

where the option exercise threshold $\hat{D}(\theta)$ is chosen optimally to maximize the value of the option. Equation (27) implies that the value of options, $V_O(\theta, D)$ must satisfy

$$\bar{r}(\theta_H) V_O(\theta_H, D) = \frac{\partial}{\partial D} V_O(\theta_H, D) \left[ \mu_D - \gamma \rho \sigma_C^2(\theta_H) \right]$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial D^2} V_O(\theta_H, D) \left[ \sigma_C^2(\theta_H) + \sigma_D^2(\theta_H) \right]$$

$$+ \lambda \left[ \omega^{-1} V_O(\theta_L, D) - V_O(\theta_H, D) \right]$$

(35)
and

\[ \tilde{r}(\theta_L) V_O(\theta_L, D) = \frac{\partial}{\partial D} V_O(\theta_L, D) D [\mu_D - \gamma \rho \sigma_C^2(\theta_L)] + \frac{1}{2} \frac{\partial^2}{\partial^2 D} V_O(\theta_L, D) \rho^2 [\sigma_C^2(\theta_L) + \sigma_D^2(\theta_L)] + \lambda [\hat{\omega} V_H(D) - V(\theta_L, D)]. \]  

(36)

We guess \( V_O(\theta, D) \) takes the form of \( b(\theta) D^\alpha \), then (35) and (36) can be written as:

\[ \tilde{r}(\theta_H) b(\theta_H) D^\alpha = \alpha b(\theta_H) D^\alpha [\mu_D - \gamma \rho \sigma_C^2(\theta_H)] + \frac{1}{2} b(\theta_H) \alpha (\alpha - 1) D^\alpha \rho^2 [\sigma_C^2(\theta_H) + \sigma_D^2(\theta_H)] + \lambda [\hat{\omega}^{-1} b(\theta_L) D^\alpha - b(\theta_H) D^\alpha] \]

and

\[ \tilde{r}(\theta_L) b(\theta_L) D^\alpha = \alpha b(\theta_L) D^\alpha [\mu_D - \gamma \rho \sigma_C^2(\theta_L)] + \frac{1}{2} b(\theta_L) \alpha (\alpha - 1) D^\alpha \rho^2 [\sigma_C^2(\theta_L) + \sigma_D^2(\theta_L)] + \lambda [\hat{\omega} b(\theta_H) D^\alpha - b(\theta_L) D^\alpha] \]

This implies that \( b(\theta) \) and \( \alpha \) have to jointly satisfy:

\[ \frac{1}{2} \rho^2 [\sigma_C^2(\theta_H) + \sigma_D^2(\theta_H)] \alpha^2 b(\theta_H) + \left[ \mu_D - \gamma \rho \sigma_C^2(\theta_H) - \frac{1}{2} \rho^2 [\sigma_C^2(\theta_H) + \sigma_D^2(\theta_H)] \right] \alpha b(\theta_H) - [\kappa + \lambda + \tilde{r}(\theta_H)] b(\theta_H) + \lambda \hat{\omega}^{-1} b(\theta_L) = 0, \]

(37)

and

\[ \frac{1}{2} \rho^2 [\sigma_C^2(\theta_L) + \sigma_D^2(\theta_L)] \alpha^2 b(\theta_L) + \left[ \mu_D - \gamma \rho \sigma_C^2(\theta_L) - \frac{1}{2} \rho^2 [\sigma_C^2(\theta_L) + \sigma_D^2(\theta_L)] \right] \alpha b(\theta_L) - [\kappa + \lambda + \tilde{r}(\theta_L)] b(\theta_L) + \lambda \hat{\omega} b(\theta_H) = 0. \]

(38)

We normalize \( b(\theta_L) = 1 \), and denote \( e = b(\theta_H) \). Then (38) and (37) together imply that \( \alpha \) has
to solve (33). The general solution is therefore of the form:

\[ V_O (\theta_H, D) = \sum_{j=1}^{4} K_j e (\alpha_j) D^{\alpha_j}; \quad V_O (\theta_L, D) = \sum_{j=1}^{4} K_j D^{\alpha_j} \]

The boundary condition \( V_O (\theta, 0) = 0 \) implies \( K_3 = K_4 = 0 \). The constants \( K_1, K_2 \) and the option exercise thresholds \( \hat{D} (\theta_H) \) and \( \hat{D} (\theta_L) \) are jointly determined by value matching and smooth pasting conditions. This completes the proof. ■

4.2 Several Useful Results

Here we establish several useful results for the general model specified in (1)-(3). This will prepare us for the proofs of of Proposition 2 and 3.

Lemma 4 Let \( e (\alpha) \) be defined as in equation (34), then \( e (\alpha_1) \in (0, 1) \) is equivalent to the following:

1. \( \hat{\omega}^{-1} f_H (\alpha_1) > \hat{\omega} f_L (\alpha_1) \).
2. \( \hat{\omega}^{-1} f_H (\alpha_1) > -\lambda \).
3. \( -\lambda > \hat{\omega} f_L (\alpha_1) \).

Proof. Using the result from Lemma 2, \( \max \{ \alpha_L^-, \alpha_H^- \} < \alpha_1 < \min \{ \alpha_L^+, \alpha_H^+ \} \). Therefore, \( f_L (\alpha_1), f_H (\alpha_1) < 0 \), and \( e (\alpha_1) > 0 \). It is therefore enough to show that \( e (\alpha_1) < 1 \) if and only if \( \hat{\omega}^{-1} f_H (\alpha_1) > \hat{\omega} f_L (\alpha_1) \).

First assume \( e (\alpha_1) < 1 \). By the definition of \( e (\alpha) \) in equation (34), this implies that \( -\hat{\omega}^{-1} f_H (\alpha_1) > \lambda \) and \( \lambda > -\hat{\omega} f_L (\alpha_1) \). Therefore \( \hat{\omega}^{-1} f_H (\alpha_1) > \hat{\omega} f_L (\alpha_1) \) follows.

Next, starting from the assumption \( \hat{\omega}^{-1} f_H (\alpha_1) > \hat{\omega} f_L (\alpha_1) \), we have \( -\hat{\omega}^{-1} f_H (\alpha_1) < -\hat{\omega} f_L (\alpha_1) \). Also, \( [-\hat{\omega}^{-1} f_H (\alpha_1)] \times [-\hat{\omega} f_L (\alpha_1)] = \lambda^2 \) and \( -\hat{\omega}^{-1} f_H (\alpha_1) > 0, -\hat{\omega} f_L (\alpha_1) > 0 \). Therefore it must be \( -\hat{\omega}^{-1} f_H (\alpha_1) \times (-\hat{\omega} f_L (\alpha_1)) = \lambda^2 \) and \( -\hat{\omega}^{-1} f_H (\alpha_1) > 0, -\hat{\omega} f_L (\alpha_1) > 0 \). Therefore it must be \( -\hat{\omega}^{-1} f_H (\alpha_1) = \lambda < -\hat{\omega} f_L (\alpha_1) \), which implies \( e (\alpha_1) < 1 \), as needed.

The equivalence between 1), 2) and 3) follows from the fact that \( f_H (\alpha) f_L (\alpha) - \lambda^2 = 0 \). ■

Using the above lemma, we prove the following necessary and sufficient condition for \( e (\alpha_1) \in (0, 1) \) in terms of the preference and technology parameters of the model.

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Lemma 5 \( e(\alpha_1) \in (0, 1) \) is equivalent to the following condition:

\[
\left[ \frac{1}{2} - \frac{\mu_D - \gamma \rho \sigma^2_C(\theta_H)}{\rho^2 [\sigma^2_C(\theta_H) + \sigma^2_D(\theta_H)]} \right]^2 + \frac{2[k + r(\theta_H)]}{\rho^2[\sigma^2_C(\theta_H) + \sigma^2_D(\theta_H)]} - \frac{\mu_D - \gamma \rho \sigma^2_C(\theta_H)}{\rho^2[\sigma^2_C(\theta_H) + \sigma^2_D(\theta_H)]} > 0 \quad (39)
\]

Proof. Let \( g_H(\alpha) = f_H(\alpha) + \lambda \hat{\omega} \) and \( g_L(\alpha) = f_L(\alpha) + \lambda \hat{\omega}^{-1} \), then by the above lemma, \( e(\alpha_1) \in (0, 1) \) is equivalent to \( g_H(\alpha_1) > 0 \) and \( g_L(\alpha_1) < 0 \). Let \( \hat{\alpha}_H^+ \) and \( \hat{\alpha}_L^+ \) be the larger root of \( g_H(\alpha) = 0 \) and that of \( g_L(\alpha) = 0 \), respectively. \( e(\alpha_1) \in (0, 1) \) is then equivalent to \( \hat{\alpha}_H^+ < \hat{\alpha}_L^+ \), which is condition (39). 

Using Lemma 3, the value matching and smooth pasting conditions are written as:

\[
K_1 e_1 \hat{D}(\theta_L)^{\alpha_1} + K_2 e_2 \hat{D}(\theta_L)^{\alpha_2} = a(\theta_L) \hat{D}(\theta_L) - 1 \quad (40)
\]

\[
\alpha_1 K_1 e_1 \hat{D}(\theta_L)^{\alpha_1 - 1} + \alpha_2 K_2 e_2 \hat{D}(\theta_L)^{\alpha_2 - 1} = a(\theta_L), \quad (41)
\]

and

\[
K_1 e_1 \hat{D}(\theta_H)^{\alpha_1} + K_2 e_2 \hat{D}(\theta_H)^{\alpha_2} = a(\theta_H) \hat{D}(\theta_H) - 1 \quad (42)
\]

\[
\alpha_1 K_1 e_1 \hat{D}(\theta_H)^{\alpha_1 - 1} + \alpha_2 K_2 e_2 \hat{D}(\theta_H)^{\alpha_2 - 1} = a(\theta_H). \quad (43)
\]

The above four equations determine \( K_1, K_2 \) and the decision rules \( \hat{D}(\theta_L), \hat{D}(\theta_H) \). The following lemma provides a characterization of the signs of the constants, \( K_1 \) and \( K_2 \).

Lemma 6 \( K_1 > 0 \) and 

\[
a(\theta) \hat{D}(\theta) > \frac{\alpha_2}{\alpha_2 - 1} \quad \text{for} \quad \theta = \theta_H, \theta_L.
\]

In addition, \( K_2 < 0 \) if and only 

\[
\frac{\alpha_2}{\alpha_2 - 1} < a(\theta_L) \hat{D}(\theta_L) < \frac{\alpha_1}{\alpha_1 - 1} < a(\theta_H) \hat{D}(\theta_H).
\]

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**Proof.** First, we express $K_1$ and $K_2$ as functions of $\dot{D}(\theta_H)$ using equations (42) and (43):

\[K_1 = \frac{(\alpha_2 - 1)}{D(\theta_H)^{\alpha_1}(\alpha_2 - \alpha_1)} \left[ a(\theta_H) \dot{D}(\theta_H) - \frac{\alpha_2}{\alpha_2 - 1} \right], \quad (44)\]

\[K_2 = \frac{- (\alpha_1 - 1)}{D(\theta_H)^{\alpha_2}(\alpha_2 - \alpha_1)} \left[ a(\theta_H) \dot{D}(\theta_H) - \frac{\alpha_1}{\alpha_1 - 1} \right]. \quad (45)\]

Given the result in lemma 2, $1 < \alpha_1 < \alpha_2$. The following four cases are mutually exclusive and collectively exhaustive. Case 1) $a(\theta_H) \dot{D}(\theta_H) < \frac{\alpha_2}{\alpha_2 - 1}$ and $K_1 < 0$, $K_2 > 0$; Case 2) $\frac{\alpha_2}{\alpha_2 - 1} < a(\theta_H) \dot{D}(\theta_H) < \frac{\alpha_1}{\alpha_1 - 1}$, and $K_1 > 0$, $K_2 < 0$; Case 3) $a(\theta_H) \dot{D}(\theta_H) > \frac{\alpha_1}{\alpha_1 - 1}$ and $K_1 > 0$, $K_2 < 0$; Case 4) Either $K_1$ or $K_2 = 0$.

Similarly, we can express $K_1$ and $K_2$ as functions of $\dot{D}(\theta_L)$ using equations (40) and (41):

\[K_1 = \frac{\alpha_2 - 1}{e_1 D(\theta_L)^{\alpha_1}(\alpha_2 - \alpha_1)} \left[ a(\theta_L) \dot{D}(\theta_L) - \frac{\alpha_2}{\alpha_2 - 1} \right], \quad (46)\]

\[K_2 = \frac{- (\alpha_1 - 1)}{e_2 D(\theta_L)^{\alpha_2}(\alpha_2 - \alpha_1)} \left[ a(\theta_L) \dot{D}(\theta_L) - \frac{\alpha_1}{\alpha_1 - 1} \right]. \quad (47)\]

Given that $e_1 > 0$ and $e_2 < 0$, one the following four mutually exclusive cases must be true. Case i) $a(\theta_L) \dot{D}(\theta_L) < \frac{\alpha_2}{\alpha_2 - 1}$ and $K_1 < 0$ and $K_2 < 0$; Case ii) $\frac{\alpha_2}{\alpha_2 - 1} < a(\theta_L) \dot{D}(\theta_L) < \frac{\alpha_1}{\alpha_1 - 1}$ and $K_1 > 0$, $K_2 < 0$; Case iii) $a(\theta_L) \dot{D}(\theta_L) > \frac{\alpha_1}{\alpha_1 - 1}$ and $K_1 > 0$, $K_2 > 0$; Case iv) Either $K_1$ or $K_2 = 0$.

Because the solution in (44)-(47) must agree with each other, case 1), case 4), case i) and case iv) can be easily ruled out. We are left with two possibilities. Case A:

\[\frac{\alpha_2}{\alpha_2 - 1} < a(\theta_H) \dot{D}(\theta_H) < \frac{\alpha_1}{\alpha_1 - 1} < a(\theta_H) \dot{D}(\theta_L), \text{ and } K_1, K_2 > 0\]

and Case B:

\[\frac{\alpha_2}{\alpha_2 - 1} < a(\theta_L) \dot{D}(\theta_L) < \frac{\alpha_1}{\alpha_1 - 1} < a(\theta_L) \dot{D}(\theta_H), \text{ and } K_1 > 0, K_2 < 0.\]

This proves the lemma. ■

### 4.3 Proof of Proposition 2

Note that in the economy with idiosyncratic volatility shocks, $\sigma_C(\theta) = \sigma$ is a constant, and $\sigma_D(\theta) = \theta$ for $\theta = \theta_H, \theta_L$. We first prove the following lemma.
Lemma 7 In the economy with idiosyncratic shocks, $0 < \epsilon_1 < 1$.

Proof. By the result of Lemma 4, it is enough to show $f_H(\alpha_1) > f_L(\alpha_1)$. Using the fact that $\sigma_C(\theta) = \sigma$ and $\sigma_D(\theta) = \theta$,

$$f_H(\alpha) - f_L(\alpha) = \frac{1}{2} \left[ \theta_H^2 - \theta_L^2 \right] \rho^2 \alpha (\alpha - 1).$$

Clearly, $f_H(\alpha_1) - f_L(\alpha_1) > 0$ because $\alpha_1 > 1$. $\blacksquare$

In the case of idiosyncratic volatility, the value function of assets in place can be simplified to $V_A(\theta, D) = aD$, where

$$a = a(\theta_L) = a(\theta_H) = \frac{1}{\kappa + \beta + \frac{1}{\psi} \mu_C + \frac{1}{2} \gamma \left( 2\rho - 1 - \frac{1}{\psi} \right) \sigma^2 - \mu_D}.$$

Let $\hat{D} = \max\left\{ \hat{D}(\theta_H), \hat{D}(\theta_L) \right\}$. We first prove the following lemma.

Lemma 8 In the economy with idiosyncratic volatility shocks, $V_O(\theta_H, D) > V_O(\theta_L, D)$ on $(0, \hat{D})$.

Proof. We first show that one of the following two cases must be true. Case 1: $V_O(\theta_H, D) > V_O(\theta_L, D)$ for all $D \in (0, \hat{D})$; Case 2: $V_O(\theta_H, D) < V_O(\theta_L, D)$ for all $D \in (0, \hat{D})$. To see this, note that $V_O(\theta_H, 0) = V_O(\theta_L, 0) = 0$. Also, value matching implies $V_O(\theta_H, \hat{D}) = V_O(\theta_L, \hat{D}) = a\hat{D}$. Because both value functions are continuously differentiable, $\frac{\partial}{\partial D} V_O(\theta_H, D)$ and $\frac{\partial}{\partial D} V_O(\theta_L, D)$ must cross at least once on $(0, \hat{D})$. To establish that either Case 1 or Case 2 must be true, it is enough to show that they can cross each other at most once on $(0, \hat{D})$.

First, let $\bar{D} = \min\left\{ \hat{D}(\theta_H), \hat{D}(\theta_L) \right\}$. $\frac{\partial}{\partial D} V_O(\theta_H, D)$ and $\frac{\partial}{\partial D} V_O(\theta_L, D)$ can not cross each other on $(\bar{D}, \hat{D})$. Assume without loss of generality, $\bar{D} = \hat{D}(\theta_H)$ and $\hat{D} = \hat{D}(\theta_L)$. Then $\frac{\partial}{\partial D} V_O(\theta_H, D) = a$ on $(\bar{D}, \hat{D})$ and $\frac{\partial}{\partial D} V_O(\theta_L, D) < a$ by smooth pasting and the fact that $V(\theta_L, D)$ is strictly convex. Therefore, in this case, $\frac{\partial}{\partial D} V_O(\theta_H, D) = \frac{\partial}{\partial D} V_O(\theta_L, D)$ cannot happen on $(\bar{D}, \hat{D})$.

On $(0, \bar{D})$,

$$\frac{\partial}{\partial D} V_O(\theta_L, D) = \alpha_1 K_1 \epsilon_1 D^{\alpha_1 - 1} + \alpha_2 K_2 \epsilon_2 D^{\alpha_2 - 1},$$

$$\frac{\partial}{\partial D} V_O(\theta_H, D) = \alpha_1 K_1 D^{\alpha_1 - 1} + \alpha_2 K_2 D^{\alpha_2 - 1}.$$
The solution to $\frac{\partial}{\partial D} V_O(\theta_L, D) = \frac{\partial}{\partial D} V_O(\theta_H, D)$ is given by

$$D = \left[ \frac{(1 - e_1) \alpha_1 K_1}{(e_2 - 1) \alpha_2 K_2} \right]^{\frac{1}{\alpha_2 - \alpha_1}},$$

which is unique.

Next, we show that Case 2 cannot be true. It is enough to show that $V_O(\theta_H, D) > V_O(\theta_L, D)$ for some $D \in (0, \hat{D})$. Consider

$$\frac{V_O(\theta_H, \varepsilon)}{V_O(\theta_D, \varepsilon)} = \frac{K_1 \varepsilon^{\alpha_1} + K_2 \varepsilon^{\alpha_2}}{K_1 e_1 \varepsilon^{\alpha_1} + K_2 e_2 \varepsilon^{\alpha_2}}.$$

For $\varepsilon \to 0$, the first terms in the numerator and denominator dominate because $1 < \alpha_1 < \alpha_2$. We have

$$\frac{V_O(\theta_H, \varepsilon)}{V_O(\theta_D, \varepsilon)} \to \frac{K_1 \varepsilon^{\alpha_1}}{K_1 e_1 \varepsilon^{\alpha_1}} = \frac{1}{e_1} > 1$$

by Lemma 7. This completes the proof. □

It remains to show that $\hat{D}(\theta_H) > \hat{D}(\theta_L)$. Note that $\frac{\partial}{\partial D} V_O(\theta_L, D) = \frac{\partial}{\partial D} V_O(\theta_H, D)$ must have a solution on $(0, \hat{D})$. Note also, $e_1 \in (0, 1)$ and $e_2 < 0$. Equation (48) therefore implies $K_2 < 0$. Using Lemma 6, we must have

$$\frac{\alpha_2}{a(\alpha_2 - 1)} < \hat{D}(\theta_L) < \frac{\alpha_1}{a(\alpha_1 - 1)} < \hat{D}(\theta_H).$$

4.4 Proof of Proposition 3

We first prove $e_1 < 1$. This is the following lemma.

**Lemma 9** Suppose

$$\kappa + \beta + \frac{1}{\psi} \mu_C - \frac{1}{2} \gamma \left(1 + \frac{1}{\psi}\right) \theta^2 > \left(1 + \frac{2\gamma}{\rho}\right) (\mu_D + \gamma \rho \sigma^2), \text{ for } \theta = \theta_H, \theta_L,$$

then $0 < e_1 < 1$ in the economy with aggregate volatility shocks.

**Proof.** Let $g_H(\alpha) = f_H(\alpha) + \lambda \hat{\omega}$ and $g_L(\alpha) = f_L(\alpha_1) + \lambda \hat{\omega}^{-1}$ and let $\hat{\alpha}_H^+$ and $\hat{\alpha}_L^+$ be the larger root of $g_H(\alpha) = 0$ and that of $g_L(\alpha) = 0$. By Lemma 5, it is enough to show $\hat{\alpha}_H^+ < \hat{\alpha}_L^+$. It is
enough to show $g_H(\alpha) > g_L(\alpha)$ at $\alpha = \hat{\alpha}_L^+$. Using (28)

\[
\begin{align*}
g_L(\alpha) &= \frac{1}{2} \gamma \tau_L^2 \alpha^2 + \left( \mu_L - \frac{1}{2} \gamma \tau_L^2 \right) \alpha - [\kappa + r(\theta_L)] \\
g_H(\alpha) &= \frac{1}{2} \gamma \tau_H^2 \alpha^2 + \left( \mu_H - \frac{1}{2} \gamma \tau_H^2 \right) \alpha - [\kappa + r(\theta_H)],
\end{align*}
\]

where $\tau_L$, $\mu_L$, $\tau_H$, and $\mu_H$ are given in equations (37)-(32), and $r(\theta)$ is the risk-free interest rate given in (28). Using (28) and (37)-(32), we have

\[
g_H(\alpha) - g_L(\alpha) = \rho \alpha \left[ \frac{1}{2} \gamma (\alpha - 1) - \gamma \right] (\theta_H^2 - \theta_L^2) + [r(\theta_L) - r(\theta_H)].
\]

Note that $r(\theta_L) > r(\theta_H)$. Therefore a sufficient condition for $g_H(\hat{\alpha}_L^+) > g_L(\hat{\alpha}_L^+)$ is $\frac{1}{2} \rho (\hat{\alpha}_L^+ - 1) - \gamma > 0$, or $\hat{\alpha}_L^+ > 1 + \frac{2\gamma}{\rho}$.

Note that

\[
\hat{\alpha}_L^+ = \left[ \frac{1}{2} - \frac{\mu_D - \gamma \rho \sigma_C^2(\theta_L)}{\rho^2 \left[ \sigma_C^2(\theta_L) + \sigma_D^2(\theta_L) \right]} \right] + \sqrt{\left[ \frac{1}{2} - \frac{\mu_D - \gamma \rho \sigma_C^2(\theta_L)}{\rho^2 \left[ \sigma_C^2(\theta_L) + \sigma_D^2(\theta_L) \right]} \right]^2 + \frac{2 [\kappa + r(\theta_L)]}{\rho^2 \left[ \sigma_C^2(\theta_L) + \sigma_D^2(\theta_L) \right]}}.
\]

It is straightforward to show that condition (49) implies $\hat{\alpha}_L^+ > 1 + \frac{2\gamma}{\rho}$. Q.E.D. 

Now we show that the conclusions of Proposition 3 are true under the condition $e_1 \in (0, 1)$. We continue to denote $\hat{D} = \max \{ \hat{D}(\theta_H), \hat{D}(\theta_L) \}$ and $\bar{D} = \min \{ \hat{D}(\theta_H), \hat{D}(\theta_L) \}$.

We first prove the existence of $D^*$. Note at $\hat{D}$, $V_O(\theta_L, D) = a(\theta_L) - 1 > V_O(\theta_H, D) = a(\theta_H) - 1$. Also, for $\varepsilon \in (0, \hat{D})$ and $\varepsilon$ small,

\[
\frac{V_O(\theta_H, \varepsilon)}{V_O(\theta_D, \varepsilon)} = \frac{K_1 \varepsilon^{\alpha_1} + K_2 \varepsilon^{\alpha_2}}{K_1 e_1 \varepsilon^{\alpha_1} + K_2 e_2 \varepsilon^{\alpha_2}} \to \frac{1}{e_1} > 1.
\]

Because both $V_O(\theta_L, D)$ and $V_O(\theta_H, D)$ are continuous, there must be a $D^* \in (0, \hat{D})$ such that $V_O(\theta_L, D^*) = V_O(\theta_H, D^*)$.

We next show that $D^*$ must be unique. First consider the case there exists $D^* \in (0, \hat{D})$ such that $V_O(\theta_L, D^*) = V_O(\theta_H, D^*)$. In this case the solution to $V_O(\theta_L, D) = V_O(\theta_H, D)$ on $(0, \hat{D})$ is

\[
D = \left[ \frac{1 - e_1}{e_2 - 1} \frac{K_1}{K_2} \right]^{\frac{1}{\alpha_2 - \alpha_1}}, \quad (50)
\]
and is unique. Furthermore, there cannot be another solution to $V_O(\theta_L, D) = V_O(\theta_H, D)$ on $(\tilde{D}, \hat{D})$. To see this, first assume $\tilde{D} = \hat{D}(\theta_L)$, $\hat{D} = \hat{D}(\theta_H)$. In this case, the fact that $V_O(\theta_L, D)$ and $V_O(\theta_H, D)$ cross only once on $(0, \tilde{D})$ and $V_O(\theta_L, \varepsilon) < V_O(\theta_H, \varepsilon)$ implies $V_O(\theta_L, \tilde{D}) \geq V_O(\theta_H, \tilde{D})$. In addition, on $(\tilde{D}, \hat{D})$, $\frac{\partial}{\partial D}V_O(\theta_L, D) = a(\theta_L) > a(\theta_H) > \frac{\partial}{\partial D}V_O(\theta_H, D)$, where the second inequality is due to the strictly convexity of the value function. This means $V_O(\theta_L, D) > V_O(\theta_H, D)$ on $(\tilde{D}, \hat{D})$. Next, assume $\tilde{D} = \hat{D}(\theta_H)$, $\hat{D} = \hat{D}(\theta_L)$. In this case, on $(\tilde{D}, \hat{D})$, $V_O(\theta_L, D) \geq a(\theta_L) D - 1 > a(\theta_H) D - 1 = V_O(\theta_H, D)$ and therefore, $V_O(\theta_L, D)$ and $V_O(\theta_H, D)$ cannot cross each other on $(\tilde{D}, \hat{D})$ either.

Next consider the case in which there does not exist $D^* \in (0, \tilde{D}]$ such that $V_O(\theta_L, D^*) = V_O(\theta_H, D^*)$. There must be at least one $D^* \in (\tilde{D}, \hat{D})$. Denote $D_{MIN}^* = \min \{ D \in (\tilde{D}, \hat{D}) : \forall \theta \in \theta, V_O(\theta_L, D) = V_O(\theta_H, D) \}$. First assume $\tilde{D} = \hat{D}(\theta_L)$, $\hat{D} = \hat{D}(\theta_H)$. Note on $(D_{MIN}^*, \tilde{D})$, $\frac{\partial}{\partial D}V_O(\theta_L, D) = a(\theta_L) > a(\theta_H) > \frac{\partial}{\partial D}V_O(\theta_H, D)$. Hence $D^*$ must be unique. Note also $\tilde{D} = \hat{D}(\theta_H)$, $\hat{D} = \hat{D}(\theta_L)$ is not possible, because on $(\tilde{D}, \hat{D})$, $V_O(\theta_L, D) \geq a(\theta_L) D - 1 > a(\theta_H) - 1 = V_O(\theta_H, D)$ and there can be no further intersections between $V_O(\theta_L, D)$ and $V_O(\theta_H, D)$, a contradiction.

To see $\hat{D}(\theta_H) > \hat{D}(\theta_L)$, it is sufficient to prove

$$\frac{\alpha_2}{\alpha_2 - 1} < a(\theta_L) \hat{D}(\theta_L) < \frac{\alpha_1}{\alpha_1 - 1} < a(\theta_H) \hat{D}(\theta_H).$$

(51)

If there exists $D^* \in (0, \tilde{D}]$ such that $V_O(\theta_L, D^*) = V_O(\theta_H, D^*)$. Equation (50) implies that $K_2 < 0$. Inequality (51) therefore follows from Lemma 6.

If not, from the above discussion, it must be $\hat{D}(\theta_L) < \hat{D}(\theta_H)$. Note at $D(\theta_L)$, $\frac{\partial}{\partial D}V_O(\theta_L, D) = a(\theta_L) > a(\theta_H) \geq \frac{\partial}{\partial D}V_O(\theta_H, D)$. Also, $\frac{\partial}{\partial D}V_O(\theta_L, \varepsilon) < \frac{\partial}{\partial D}V_O(\theta_H, \varepsilon)$ for some $\varepsilon \in (0, \hat{D})$ and $\varepsilon$ close to 0. Because both $V_O(\theta_L, D)$ and $V_O(\theta_H, D)$ are continuously differentiable, there must exist $D^{**} \in (\theta_L, \hat{D}(\theta_L))$ such that $\frac{\partial}{\partial D}V_O(\theta_L, D^{**}) = \frac{\partial}{\partial D}V_O(\theta_H, D^{**})$. The solution of $D^{**}$ is given in (48). Again, this implies $K_2 < 0$. As a result, (51) must be true due to Lemma 6.

To complete the proof of Proposition 3, note that under the assumption $\rho = 1$, condition (11) and (49) are equivalent.
Table 1
Parameter Configuration of the Example Economies

<table>
<thead>
<tr>
<th>Preference parameters</th>
<th>Value (dimensionless)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.02</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>2.0</td>
</tr>
<tr>
<td>$\psi$</td>
<td>2.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Consumption/cash flow</th>
<th>Value (dimensionless)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_C$</td>
<td>0.02</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.01</td>
</tr>
<tr>
<td>$\theta_H$</td>
<td>0.03</td>
</tr>
<tr>
<td>$\theta_L$</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.02</td>
</tr>
<tr>
<td>$\mu_D$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.12</td>
</tr>
<tr>
<td>$\rho$</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Table 1 presents the parameter configuration used in numerical examples of the economies with time-varying idiosyncratic volatility and time-varying aggregate volatility.
Table 2
Sorting on Exposure to Idiosyncratic Volatility

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\beta^{ID}$</th>
<th>Tobin-Q</th>
<th>R&amp;D/Assets</th>
<th>Return</th>
<th>$\Delta$Sales</th>
<th>$\Delta$I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>-0.030</td>
<td>1.11</td>
<td>3.62</td>
<td>7.48</td>
<td>2.57</td>
<td>-0.65</td>
</tr>
<tr>
<td>2</td>
<td>0.004</td>
<td>1.10</td>
<td>2.56</td>
<td>6.04</td>
<td>3.59</td>
<td>0.89</td>
</tr>
<tr>
<td>3</td>
<td>0.028</td>
<td>1.08</td>
<td>2.55</td>
<td>7.63</td>
<td>3.93</td>
<td>4.35</td>
</tr>
<tr>
<td>4</td>
<td>0.054</td>
<td>1.10</td>
<td>2.96</td>
<td>7.13</td>
<td>4.22</td>
<td>5.06</td>
</tr>
<tr>
<td>High</td>
<td>0.102</td>
<td>1.32</td>
<td>5.47</td>
<td>5.29</td>
<td>4.67</td>
<td>8.05</td>
</tr>
</tbody>
</table>

Table 2 presents characteristics of portfolios sorted on exposure to idiosyncratic volatility. Idiosyncratic volatility beta ($\beta^{ID}$), Tobin’s Q, and the ratio of R&D expenditure to lagged assets are computed at the time when portfolios are formed. Average return (Return), sales growth ($\Delta$Sales) and investment growth ($\Delta$I) are computed over the one-year holding period. Portfolios are value-weighted and rebalanced annually; the entries correspond to time-series averages over the 1964-2012 sample period; R&D/Assets, Return, $\Delta$Sales, and $\Delta$I are expressed in percentages.
Table 3
Double Sort on BM and Exposure to Idiosyncratic Volatility

**Panel A: Portfolio Characteristics at Formation**

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>BM</th>
<th>$\beta^{ID}$</th>
<th>R&amp;D/Assets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low BM</td>
<td>High BM</td>
<td>Low BM</td>
</tr>
<tr>
<td>Low $\beta^{ID}$</td>
<td>0.36</td>
<td>1.43</td>
<td>-0.016</td>
</tr>
<tr>
<td>High $\beta^{ID}$</td>
<td>0.34</td>
<td>1.41</td>
<td>0.084</td>
</tr>
</tbody>
</table>

**Panel B: Portfolio Characteristics over the Holding Period**

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Return</th>
<th>$\Delta$Sales</th>
<th>$\Delta$I</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low BM</td>
<td>High BM</td>
<td>Low BM</td>
</tr>
<tr>
<td>Low $\beta^{ID}$</td>
<td>6.24</td>
<td>10.73</td>
<td>6.05</td>
</tr>
<tr>
<td>High $\beta^{ID}$</td>
<td>3.76</td>
<td>10.86</td>
<td>11.13</td>
</tr>
</tbody>
</table>

Table 3 shows characteristics of portfolios sorted on book-to-market ratio (BM) and exposure to idiosyncratic volatility ($\beta^{ID}$). Portfolios are constructed by first sorting firms into three book-to-market portfolios, and then dividing each BM bin into three idiosyncratic volatility beta portfolios. The table presents four portfolios with opposite characteristics: low and high BM, and low and high $\beta^{ID}$. In Panel A, book-to-market ratio, $\beta^{ID}$ and R&D to lagged assets are computed at the time when portfolios are formed. In Panel B, average return (Return), sales growth ($\Delta$Sales) and growth in investment ($\Delta$I) are computed over the one-year holding period. Portfolios are value-weighted and rebalanced annually; the entries correspond to time-series averages over the 1964-2012 period; R&D/Assets, Return, $\Delta$Sales, and $\Delta$I are expressed in percentages.
Table 4
Idiosyncratic Volatility Exposure and Future Investment Growth

<table>
<thead>
<tr>
<th></th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
<th>Model IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Using $\beta^{ID}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.87</td>
<td>1.49</td>
<td>0.55</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>(4.52)</td>
<td>(7.59)</td>
<td>(4.90)</td>
<td>(6.58)</td>
</tr>
<tr>
<td>Panel B: Using Portfolio Position</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>0.06</td>
<td>0.06</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td>(6.85)</td>
<td>(6.92)</td>
<td>(3.57)</td>
<td>(3.58)</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>0.09</td>
<td>0.10</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>(10.69)</td>
<td>(11.21)</td>
<td>(5.36)</td>
<td>(5.76)</td>
</tr>
<tr>
<td>$\phi_4$</td>
<td>0.11</td>
<td>0.14</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>(9.75)</td>
<td>(13.29)</td>
<td>(6.80)</td>
<td>(6.69)</td>
</tr>
<tr>
<td>$\phi_5$</td>
<td>0.14</td>
<td>0.22</td>
<td>0.09</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>(7.71)</td>
<td>(16.44)</td>
<td>(7.23)</td>
<td>(8.20)</td>
</tr>
</tbody>
</table>

| Firm FE | No | Yes | No | Yes |
| Controls | No | No | Yes | Yes |

Table 4 documents predictability of firms’ investment growth rate, $\log\left(\frac{I_{t+1}}{I_t}\right)$, by their exposure to idiosyncratic volatility. In Panel A, $\phi$ is the regression coefficient on the idiosyncratic volatility beta ($\beta^{ID}$); in Panel B, $\phi_j$ is the slope coefficient on a dummy variable that equals one if a firm belongs to the $j$-quintile portfolio of firms sorted on $\beta^{ID}$. The four columns, “Model I” through “Model IV”, correspond to different regression specifications with and without firm fixed effects, and with and without controls. The set of controls comprises the ratios of sales to assets, cash to capital, book to market, and investment to capital, Tobin’s Q, market share and firm return. Numbers in parentheses are t-statistics based on standard errors clustered by firm and time. The data span the 1964-2012 period.
Table 5
Idiosyncratic Volatility Exposure, Future Investment and Returns

<table>
<thead>
<tr>
<th></th>
<th>ΔI</th>
<th></th>
<th>I/K</th>
<th></th>
<th>Return</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>t-stat</td>
<td>Estimate</td>
<td>t-stat</td>
<td>Estimate</td>
<td>t-stat</td>
</tr>
<tr>
<td>1-year</td>
<td>0.73</td>
<td>6.58</td>
<td>0.61</td>
<td>5.76</td>
<td>-0.41</td>
<td>-1.89</td>
</tr>
<tr>
<td>2-year</td>
<td>0.62</td>
<td>5.19</td>
<td>0.51</td>
<td>4.33</td>
<td>-0.21</td>
<td>-1.80</td>
</tr>
<tr>
<td>3-year</td>
<td>0.61</td>
<td>5.74</td>
<td>0.48</td>
<td>4.55</td>
<td>-0.12</td>
<td>-2.14</td>
</tr>
</tbody>
</table>

Table 5 documents predictability of the (annualized) cumulative 1-, 2-, and 3-year ahead investment growth (ΔI), investment rate (I/K) and return (Return) by firms’ exposure to idiosyncratic volatility. Regression specifications include firm fixed effects and control for firm characteristics. The return regression controls for firms’ sales-to-assets and book-to-market ratios, investment rate and market capitalization. In investment regressions, the set of controls is augmented by the ratio of cash to capital, Tobin’s Q and firm return. T-statistics are based on standard errors clustered by firm and time. The data span the 1964-2012 period.
<table>
<thead>
<tr>
<th>Portfolio</th>
<th>ΔSales</th>
<th>ΔI</th>
<th>ΔSales</th>
<th>ΔI</th>
<th>ΔSales</th>
<th>ΔI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>2.48</td>
<td>-0.51</td>
<td>2.45</td>
<td>-0.75</td>
<td>3.08</td>
<td>0.82</td>
</tr>
<tr>
<td>2</td>
<td>3.61</td>
<td>1.56</td>
<td>3.60</td>
<td>1.19</td>
<td>3.03</td>
<td>1.27</td>
</tr>
<tr>
<td>3</td>
<td>3.73</td>
<td>2.72</td>
<td>3.86</td>
<td>3.10</td>
<td>3.72</td>
<td>3.10</td>
</tr>
<tr>
<td>4</td>
<td>4.80</td>
<td>6.05</td>
<td>4.86</td>
<td>5.50</td>
<td>4.07</td>
<td>5.17</td>
</tr>
<tr>
<td>High</td>
<td>5.31</td>
<td>9.89</td>
<td>5.21</td>
<td>8.02</td>
<td>5.53</td>
<td>7.52</td>
</tr>
</tbody>
</table>

Table 6 reports average sales growth ($\Delta$Sales) and investment growth ($\Delta$I) of portfolios sorted on exposure to idiosyncratic volatility. In the first set of columns (FF-Adjusted), idiosyncratic volatility is constructed by adjusting firm returns using the Fama-French three-factor model. In the other columns, idiosyncratic volatility betas are measures using either a 2-year or a 5-year estimation window. Portfolios are value-weighted and rebalanced annually; the entries correspond to time-series averages over the 1964-2012 sample period and are expressed in percentages.
Table 7  
Double Sort on ID-Volatility Level and ID-Volatility Exposure

**Panel A: Conditional Sort**

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Return</th>
<th>ΔI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low ID</td>
<td>Med ID</td>
</tr>
<tr>
<td>Low $\beta^{ID}$</td>
<td>6.58</td>
<td>8.97</td>
</tr>
<tr>
<td>Med $\beta^{ID}$</td>
<td>6.76</td>
<td>8.04</td>
</tr>
<tr>
<td>High $\beta^{ID}$</td>
<td>6.87</td>
<td>7.34</td>
</tr>
</tbody>
</table>

**Panel B: Independent Sort**

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Return</th>
<th>ΔI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low ID</td>
<td>Med ID</td>
</tr>
<tr>
<td>Low $\beta^{ID}$</td>
<td>6.54</td>
<td>8.61</td>
</tr>
<tr>
<td>Med $\beta^{ID}$</td>
<td>7.07</td>
<td>7.87</td>
</tr>
<tr>
<td>High $\beta^{ID}$</td>
<td>7.10</td>
<td>7.51</td>
</tr>
</tbody>
</table>

Table 7 shows characteristics of $3 \times 3$ portfolios sorted on the level of idiosyncratic volatility (ID) and exposure to idiosyncratic volatility ($\beta^{ID}$). In Panel A, we first rank firms on the level of idiosyncratic volatility, and then divide the resulting portfolios into three $\beta^{ID}$ bins. In Panel B, portfolios are intersection of two independent sorts. Average return (Return) and growth in investment (ΔI) are computed over the one-year holding period and are expressed in percentages. Portfolios are value-weighted and rebalanced annually; the entries correspond to time-series averages over the 1964-2012 period.
Table 8
Sorting on Exposure to Aggregate Volatility

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\beta^A$</th>
<th>Tobin-Q</th>
<th>R&amp;D/Assets</th>
<th>Return</th>
<th>$\Delta$Sales</th>
<th>$\Delta$I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>-0.093</td>
<td>1.22</td>
<td>5.14</td>
<td>8.87</td>
<td>4.01</td>
<td>0.67</td>
</tr>
<tr>
<td>2</td>
<td>-0.056</td>
<td>1.11</td>
<td>3.44</td>
<td>8.80</td>
<td>3.47</td>
<td>2.77</td>
</tr>
<tr>
<td>3</td>
<td>-0.036</td>
<td>1.10</td>
<td>2.71</td>
<td>7.84</td>
<td>3.07</td>
<td>1.83</td>
</tr>
<tr>
<td>4</td>
<td>-0.018</td>
<td>1.11</td>
<td>2.65</td>
<td>6.20</td>
<td>3.38</td>
<td>2.26</td>
</tr>
<tr>
<td>High</td>
<td>0.006</td>
<td>1.15</td>
<td>3.15</td>
<td>4.95</td>
<td>4.47</td>
<td>2.58</td>
</tr>
</tbody>
</table>

Table 8 presents characteristics of portfolios sorted on exposure to aggregate volatility. Aggregate volatility beta ($\beta^A$), Tobin’s Q, and the ratio of R&D expenditure to lagged assets are computed at the time when portfolios are formed. Average return (Return), sales growth ($\Delta$Sales) and investment growth ($\Delta$I) are computed over the one-year holding period. Portfolios are value-weighted and rebalanced annually; the entries correspond to time-series averages over the 1964-2012 sample period; R&D/Assets, Return, $\Delta$Sales, and $\Delta$I are expressed in percentages.
### Table 9

**Aggregate Volatility Exposure and Future Investment Growth**

<table>
<thead>
<tr>
<th></th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
<th>Model IV</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Using $\beta^A$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.415</td>
<td>0.482</td>
<td>0.304</td>
<td>0.256</td>
</tr>
<tr>
<td></td>
<td>(1.25)</td>
<td>(1.79)</td>
<td>(1.77)</td>
<td>(1.50)</td>
</tr>
<tr>
<td><strong>Panel B: Using Portfolio Position</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>0.021</td>
<td>-0.003</td>
<td>-0.001</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>(1.60)</td>
<td>(-0.28)</td>
<td>(-0.13)</td>
<td>(0.32)</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>0.030</td>
<td>-0.002</td>
<td>-0.004</td>
<td>0.011</td>
</tr>
<tr>
<td></td>
<td>(1.69)</td>
<td>(-0.13)</td>
<td>(-0.49)</td>
<td>(1.40)</td>
</tr>
<tr>
<td>$\phi_4$</td>
<td>0.035</td>
<td>0.007</td>
<td>-0.009</td>
<td>0.008</td>
</tr>
<tr>
<td></td>
<td>(1.81)</td>
<td>(0.51)</td>
<td>(-1.03)</td>
<td>(0.88)</td>
</tr>
<tr>
<td>$\phi_5$</td>
<td>0.026</td>
<td>0.016</td>
<td>0.003</td>
<td>0.013</td>
</tr>
<tr>
<td></td>
<td>(1.10)</td>
<td>(0.90)</td>
<td>(0.30)</td>
<td>(1.24)</td>
</tr>
<tr>
<td>Firm FE</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Controls</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 9 documents predictability of firms’ investment growth rate, $\log \left( \frac{I_{t+1}}{I_t} \right)$, by their exposure to aggregate volatility. In Panel A, $\phi$ is the regression coefficient on the aggregate volatility beta ($\beta^A$); In Panel B, $\phi_j$ is the slope coefficient on a dummy variable that equals one if a firm belongs to the $j$-quintile portfolio of firms sorted on $\beta^A$. The four columns, “Model I” through “Model IV”, correspond to different regression specifications with and without firm fixed effects, and with and without controls. The set of controls comprises the ratios of sales to assets, cash to capital, book to market, and investment to capital, Tobin’s Q, market share and firm return. Numbers in parentheses are t-statistics based on standard errors clustered by firm and time. The data span the 1964-2012 period.
Table 10
Double Sort on Exposure to Volatility

Panel A: Portfolio Characteristics at Formation

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\beta^A$</th>
<th>$\beta^{ID}$</th>
<th>R&amp;D/Assets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low $\beta^A$</td>
<td>High $\beta^A$</td>
<td>Low $\beta^{ID}$</td>
</tr>
<tr>
<td>Low $\beta^{ID}$</td>
<td>-0.072</td>
<td>-0.003</td>
<td>-0.023</td>
</tr>
<tr>
<td>High $\beta^{ID}$</td>
<td>-0.080</td>
<td>-0.003</td>
<td>0.085</td>
</tr>
</tbody>
</table>

Panel B: Portfolio Characteristics over the Holding Period

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Return</th>
<th>$\Delta$Sales</th>
<th>$\Delta$I</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low $\beta^A$</td>
<td>High $\beta^A$</td>
<td>Low $\beta^{ID}$</td>
</tr>
<tr>
<td>Low $\beta^{ID}$</td>
<td>10.08</td>
<td>5.04</td>
<td>3.20</td>
</tr>
<tr>
<td>High $\beta^{ID}$</td>
<td>7.83</td>
<td>4.22</td>
<td>3.94</td>
</tr>
</tbody>
</table>

Table 10 shows characteristics of portfolios sorted on exposure to aggregate and idiosyncratic volatility ($\beta^A$ and $\beta^{ID}$, respectively). The four portfolios presented in the table correspond to portfolios with opposite characteristics in the $3 \times 3$ sort. In Panel A, $\beta^A$, $\beta^{ID}$ and R&D to lagged assets are computed at the time when portfolios are formed. In Panel B, average return (Return), sales growth ($\Delta$Sales) and growth in investment ($\Delta$I) are computed over the one-year holding period. Portfolios are value-weighted and rebalanced annually; the entries correspond to time-series averages over the 1964-2012 period; R&D/Assets, Return, $\Delta$Sales, and $\Delta$I are expressed in percentages.
Figure 1. Value of Growth Options: The Case of Idiosyncratic Volatility

Figure 1 plots the value of growth options as a function of the current level of dividends in the low idiosyncratic volatility state (dashed line) and that in the high idiosyncratic volatility state (solid line). The figure corresponds to the economy with time-varying idiosyncratic volatility.
Figure 2. Exposure to Idiosyncratic Volatility of Assets in Place and Growth Options

Figure 2 plots the change in the value of assets in place (solid line) and growth options (dashed line) as the economy shifts from the low to the high volatility state. The figure corresponds to the economy with time-varying idiosyncratic volatility.
Figure 3. Value of Growth Options: The Case of Aggregate Volatility

Figure 3 plots the value of growth options as a function of the current level of dividends in the low aggregate volatility state (dashed line) and that in the high aggregate volatility state (solid line). The figure corresponds to the economy with time-varying aggregate volatility.
Figure 4. Exposure to Aggregate Volatility of Assets in Place and Growth Options

Figure 4 plots the change in the value of assets in place (solid line) and growth options (dashed line) as the economy shifts from the low to the high volatility state. The figure corresponds to the economy with time-varying aggregate volatility.